Econ 8100 - Midterm Exam

October 19, 2020

1. A consumer faces prices $p_1 = p_2 = p_3 = 1$. This consumer must consume non-negative amounts of each good and has the utility function:

$$U(x_1, x_2, x_3) = (x_1 - 1)(x_2)(x_3 + 1)$$

A) Prove this consumer's preferences do not meet the following property:

$$\textit{Homotheticity: } x \sim x' \Rightarrow tx \sim tx' \, \forall t > 0$$

Pick a bundle like $x = (2, 2, 0)$ and $x' = (3, 1, 0)$ and $t = 2$

$$(2 - 1)(2)(1) = (3 - 1)(1)(1)$$

However:

$$(4 - 1)(4)(1) \neq (6 - 1)(2)(1)$$

Proving some implication of homothetic preferences is not met is acceptable as well.

For parts B-E, assume $m > 1$ and make sure your answers account for different conditions relating to m.

B) What are the Mashallian demands for x_1, x_2, x_3?

For $m \geq 3$:

$$x_1 = \frac{m}{3} + 1, x_2 = \frac{m}{3}, x_3 = \frac{m}{3} - 1$$

For $m < 3$

$$x_1 = \frac{m}{2} + \frac{1}{2}, x_2 = \frac{m}{2} - \frac{1}{2}$$

C) Write down an expression for the value of the multiplier on the budget constraint in the consumer's Lagrangian function for this problem. Interpret this multiplier and the expression.

For $m \geq 3$:

$$\left(\frac{m}{3}\right)^2$$

For $m < 3$

$$\left(\frac{m}{2} - \frac{1}{2}\right)$$

These are the marginal increase in utility if the budget constraint is relaxed marginally (money added to budget). Since the price of each good is 1, this value is simply the marginal utility of every good which is actually being purchased. The expressions above are the marginal utilities evaluated at the optimum.

D) Write down the indirect utility function.

For $m \geq 3$:

$$\left(\frac{m}{3}\right)^3$$

For $m < 3$

$$\left(\frac{m}{2} - \frac{1}{2}\right)^2$$

E) What is the elasticity of demand with respect to income for each good? Interpret the values. Under what conditions are they greater than, equal to, or less than one?

For $m > 3$

$$\frac{\partial\left(\frac{m}{3}+1\right)}{\partial m}\frac{m}{\frac{m}{3}+1} = \frac{m}{m+3} < 1$$

$$\frac{\partial\left(\frac{m}{3}\right)}{\partial m}\frac{m}{\frac{m}{3}} = 1$$

$$\frac{\partial\left(\frac{m}{3}-1\right)}{\partial m}\frac{m}{\frac{m}{3}-1} = \frac{m}{m-3} > 1$$

For $m < 3$

$$\frac{\partial\left(\frac{m}{2}+\frac{1}{2}\right)}{\partial m}\frac{m}{\frac{m}{2}+\frac{1}{2}} = \frac{m}{m+1} < 1$$

$$\frac{\partial\left(\frac{m}{2}-\frac{1}{2}\right)}{\partial m}\frac{m}{\frac{m}{2}-\frac{1}{2}} = \frac{m}{m-1} > 1$$

The elasticity depends on the income range. If $m > 3$ then good 2 has an elasticity of 1. That is, a 1% increase in income increases demand by 1%. However, good 1 increases by less than 1% and good 3 by more than 1%. On the other hand when $m < 3$, good 1 is elastic and good 2 is inelastic with respect to income.

F) What happens in this problem if $m < 1$?

Preferences are not monotonic. Since there is not enough to afford $x_1 \geq 0$, utility is decreasing in x_2 and x_3. Thus, it is optimal to consume any bundle with $x_2 = 0$ for any $m < 1$.

2. Discuss, in a few paragraphs how the consumer from the previous problem behaves in relation to a consumer with utility function $U(x_1, x_2, x_3) = x_1 x_2 x_3$.

This consumer always buys $\frac{m}{3}$ of each good. By comparison when $m \geq 3$, the previous consumer always buys one more x_1 and one less x_3. Thus, we can think of the additive terms on the utility of scaling up (in the case of a negative term) or down (in the case of a positive term) linearly. This interacts with the non-negativity constraint for x_3 for low income levels, distorting the relationship between the two consumer's incomes for low income levels. It is further distorted for *very* low income levels as the first consumer's preference become non-monotonic.

3. A consumer with locally non-satiated and strictly convex preferences has the following expenditure function:

$$e(p_1, p_2, u) = u(p_1^\alpha + p_2^\alpha)^{\frac{1}{\alpha}}$$

A) Find the consumer's Indirect Utility function.

$$\frac{m}{(p_1^\alpha + p_2^\alpha)^{\frac{1}{\alpha}}} = v(p_1, p_2, m)$$

B) Find the consumer's Marshallian Demands.

$$x_1(p_1, p_2, m) = -\frac{\frac{\partial\left(\frac{m}{\left(p_1^\alpha + p_2^\alpha\right)^{\frac{1}{\alpha}}}\right)}{\partial p_1}}{\frac{\partial\left(\frac{m}{\left(p_1^\alpha + p_2^\alpha\right)^{\frac{1}{\alpha}}}\right)}{\partial m}} = \frac{mp_1^{\alpha-1}}{p_1^\alpha + p_2^\alpha}$$

$$x_2(p_1, p_2, m) = -\frac{\frac{\partial\left(\frac{m}{\left(p_1^\alpha + p_2^\alpha\right)^{\frac{1}{\alpha}}}\right)}{\partial p_2}}{\frac{\partial\left(\frac{m}{\left(p_1^\alpha + p_2^\alpha\right)^{\frac{1}{\alpha}}}\right)}{\partial m}} = \frac{mp_2^{\alpha-1}}{p_1^\alpha + p_2^\alpha}$$

C) Show that the above expenditure function with $\alpha = 2$ is not a valid expenditure function because it violates some property of true expenditure functions.

$$e(p_1, p_2, u) = u\left(p_1^2 + p_2^2\right)^{\frac{1}{2}}$$

This function of not concave in prices. There are a few ways to show this. For one, it is a quasi-convex function that is homogeneous of degree 1, making it a convex function. Since it is not linear, it is thus non-concave. More directly, moving along in the direction of p_1, the second derivative of e with respect to p_1 is:

$$\frac{u}{\sqrt{p_1^2 + p_2^2}} - \frac{p_1^2 u}{(p_1^2 + p_2^2)^{3/2}} > 0$$

$$p_1^2 + p_2^2 > p_1^2$$

D) What does this violation have to do with the substitution effect?

The second derivative of e with respect to p_1 is the first derivative of the Hicksian demand for x_1 with respect to p_1. By the Slutsky equation, the substitution effect is $\frac{\partial x_1^h}{\partial p_1}$. This has to be negative by the law of demand. However, in the case of $\alpha = 2$, this value is positive.