Example: Non-Negativity Constraints and Complementary Slackness

$$u = \log(x_1) + \sqrt{x_2} + x_3$$

$$x_1 + x_2 + x_3 \leq m$$

Let's set up the Lagrangian function while putting the non-negativity constraints in explicitly:

$$\log(x_1) + \sqrt{x_2} + x_3 - \lambda(x_1 + x_2 + x_3 - m) - \mu_1(-x_1) - \mu_2(-x_2) - \mu_3(-x_3)$$

The first order conditions are:

$$\mu_1 + \frac{1}{x_1} = \lambda$$

$$\mu_2 + \frac{1}{2\sqrt{x_2}} = \lambda$$

$$\mu_3 + 1 = \lambda$$

Suppose none of our non-negativity constraints bind. By complementary slackness: $\mu_1, \mu_2, \mu_3 = 0$. The first order conditions become:

$$\frac{1}{x_1} = \lambda$$

$$\frac{1}{2\sqrt{x_2}} = \lambda$$

$$1 = \lambda$$

Solving these for x_1, x_2, x_3 :

$$1 = x_1$$

$$\frac{1}{4} = x_2$$

$$x_3 = m - \frac{5}{4}$$

If $m \geq \frac{5}{4}$, this is a valid solution to the problem and since the utility function is concave, it is sufficient for the optimum. However, note that if $m < \frac{5}{4}$, this is not a feasible solution since it violates the non-negativity constraint for x_3. What is the optimal solution in that case?

Lets suppose $x_1 \geq 0$ is binding. Thus, $x_1 = 0$. The first order condition on x_1 requires $\frac{1}{x_1} = \lambda - \mu_1$. However, at $x_1 = 0$ this equation cannot hold. Thus, $x_1 > 0$ in any solution. Similarly, we can show that $x_2 \geq 0$ cannot bind since it's first order condition requires $\frac{1}{2\sqrt{x_2}} = \lambda - \mu_2$ which is not true at $x_2 = 0$. The only alternative is that $x_1 > 0$, $x_2 > 0$ and $x_3 = 0$. Since the non-negativity constraints on x_1 and x_2 do not bind, $\mu_1 = \mu_2 = 0$. The first order conditions are:

$$\frac{1}{x_1} = \lambda$$

$$\frac{1}{2\sqrt{x_2}} = \lambda$$

$$x_1 = 2\sqrt{x_2}$$

Solving these:

$$x_1 = 2\left(\sqrt{m+1} - 1\right), x_2 = m - 2\sqrt{m+1} + 2$$

$$x_3 = 0$$

Let's look at the multipliers:

$$\lambda = \frac{1}{2\left(\sqrt{m+1} - 1\right)}$$

$$\mu_3 = \frac{1}{2\left(\sqrt{m+1} - 1\right)} - 1$$

Let's suppose $m = 1$ which meets the condition:

$$\lambda \approx 1.20711$$

$$\mu_3 = 0.20711$$

The rate at which utility increases if you relax the budget constraint is about 1.20711 (it is 1 if $m > \frac{5}{4}$). Why is it that the utility increases at a rate one less than this if we relax the non-negativity constraint on x_3. If it is relaxed we can decrease x_3 which in-turn relaxes the budget constraint at a rate of one (the price of x_3). This allows us to increase utility at a rate of 1.20711 but decreasing x_3 decreases utility at a rate of 1 so the net effect is 0.20711.

Example: Two Constraints

$$u = x_1 x_2$$

Suppose there are two budgets:

$$\frac{1}{2}x_1 + 2x_2 \leq 100$$

$$3x_1 + 2x_2 \leq 250$$

Would you rather add \$1 to budget 1 or \$1 to budget 2?

$$x_1 x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)$$

$$\frac{\partial\left(x_1 x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)\right)}{\partial x_1} = -\frac{\lambda}{2} - 3\mu + x_2$$

$$\frac{\partial\left(x_1 x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)\right)}{\partial x_2} = -2\lambda - 2\mu + x_1$$

$$x_2 = \frac{\lambda}{2} + 3\mu$$

$$x_1 = 2\lambda + 2\mu$$

$$x_1 = 60, x_2 = 35$$

$$\lambda = 22, \mu = 8$$

If budget one is relaxed, consumer can add utility at a rate of 22 while the rate is only 8 for budget 2. Budget one is more constraining. Think about why this is. Try to do the same problem with an OR constraint. That is, only one of the two constraints need to hold.