Decisions Under Uncertainty

Outcomes:

$$A \equiv \{a_1, ..., a_n\}$$

Simple gambles:

$$\mathcal{G}_s \equiv \left\{ (p_1 \circ a_1, p_2 \circ a_2, ..., a_n \circ p_n) \,|\, \sum p_i = 1 \right\}$$

Let $g, g', g'' \in \mathcal{G}_s$

Axiom 1 *Complete:*

Axiom 2 *Transitive:*

Extend $\succsim$ over A such that $a_1 \succsim a_2 \Leftrightarrow (1 \circ a_1) \succsim (1 \circ a_2)$. WLOG assume $a_i \succsim a_{i+1}$.

Axiom 3 *Continuous:* For all g $\exists p \in [0, 1]$ such that $g \sim (p \circ a_1, (1 - p) \circ a_n)$ Why is this called continuity? What sets are closed? We actually need one more thing.

Axiom 4 *Monotonic:* For all $(\alpha \circ a_1, (1 - \alpha) \circ a_n) \succsim (\beta \circ a_1, (1 - \beta) \circ a_n)$ iff $\alpha \geq \beta$,

$$\{p \in [0, 1] \,| g \succsim (p \circ a_1, (1 - p) \circ a_n)\}$$

$$\{p \in [0, 1] \,| g \precsim (p \circ a_1, (1 - p) \circ a_n)\}$$

By 3 there is some $\tilde{p}$ such that $(\tilde{p} \circ a_1, (1 - \tilde{p}) \circ a_n) \in \succsim (g), \precsim (g)$
By 4, for all $p \geq \tilde{p}$,$(p \circ a_1, (1 - p) \circ a_n) \in \succsim (g)$.

$$[\tilde{p}, 1] = \{p \in [0, 1] \,| g \succsim (p \circ a_1, (1 - p) \circ a_n)\}$$

$$[0, \tilde{p}] = \{p \in [0, 1] \,| g \precsim (p \circ a_1, (1 - p) \circ a_n)\}$$

So it is really 3 and 4 that jointly imply some sort of continuity.

Axiom 5 *Substitution:* If $g = (p_1 \circ g_1, ..., p_k \circ g_k)$ and $h\,(p_1 \circ h_1, ..., p_k \circ h_k)$ and if $g_i \sim h_i$ for all $i \in \{1, ..., k\}$.

Axiom 6 *Reduction:* For any gamble g and the simple gamble it induces g_s, $g \sim g_s$.

Under assumptions $1 - 4$, we have a utility representation.

$$u(g) : g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n)$$

Can we prove

$$g \succsim g' \Leftrightarrow u(g) \geq u(g')$$

By monotonicity,

$$u(g) \geq u(g') \Leftrightarrow (u(g) \circ a_1, (1 - u(g)) \circ a_n) \succsim (u(g') \circ a_1, (1 - u(g')) \circ a_n)$$

By continuity this

$$g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n) \succsim (u(g') \circ a_1, (1 - u(g')) \circ a_n) \sim g'$$

By transitivity:

$$(u(g) \circ a_1, (1 - u(g)) \circ a_n) \succsim (u(g') \circ a_1, (1 - u(g')) \circ a_n) \Leftrightarrow g \succsim g'$$

Expected Utility Representation

A very common utility function for decisions under uncertainty is the VNM utility function. It takes the following form:

Let u be the individual's utility function for sure outcomes. A utility function has the *expected utility* property if, letting p_i being the probability of a_i in the simple gamble generated by the gamble g:

$$V(g) = \sum_{i=1}^{n} p_i v(a_i)$$

When does a preference ordering on $\mathcal{G}$ have a representation with this property? This requires (in addition) *Substitution* and *Reduction:*

Proof:

By *monotonicity*, *continuity* (Archemedian), and *transitivity* there is a utility representation:

$$u(g) = (u(g)\, a_1, (1 - u(g))\, a_n) \sim g$$

We want to show that this utility construction is of the expected utility form:

$$u(g) = \sum_{i=1}^{n} p_i v(a_i)$$

From construction of this utility function:

$$a_i \sim (u(a_i)\, a_1, (1 - u(a_i))\, a_n) \equiv q_i$$

That is, q_i is the simple gamble over the best and worst outcome that is indifferent to sure outcome a_i.

By reduction $g \sim g_s$ where g_s is

$$\text{simple gamble induced by } g\text{: } g_s = (p_1 a_1, p_2 a_2, ..., p_n a_n)$$

By subsititution, this simple gamble is indifferent to a compound gamble replacing each sure outcome a_i with the gamble q_i H

$$g_s = (p_1 a_1, p_2 a_2, ..., p_n a_n) \sim (p_1 q_1, p_2 q_2, ..., p_n q_n) = g'$$

Now let's look at the simple gamble induced by g'. Note that g' only involves the best and worst outcomes. The induced simple gamble is:

$$g_s^{'} = \left(\sum_{i=1}^{n} p_i u(a_i) \circ a_1, \left(1 - \sum_{i=1}^{n} p_i u(a_i) \right) \circ a_n \right)$$

To recall, by reduction and substitution we have:

$$g \sim g_s \sim g' \sim g_s'$$

And thus:

$$\sum_{i=1}^{n} p_i u(a_i) = u(g_s') = u(g)$$

By this result, under substitution and reduction, there is a VNM representation of utility.

Transformations

A generic monotone transformation of u will still represent the preferences but may not have the expected utility property.

$$u(g) = \left(\sum_{i=1}^{n} p_i u(a_i) \right)^2$$

This still represents preferences but it does not have the expected utility property. The propertey is only preserved by affine transformations:

$$v(g) = \alpha + \beta u(g)$$

This works because it maintains linearity in the probabilities.

VNM Utility and Monotonic Transformations

Suppose outcomes are amount of money. $u(g) = \sum p_i v(w_i)$.

Consider gambles with the following outcomes: \$100, \$50, \$0.

$$v(\$100) = 1$$

$$v(\$50) = .75$$

$$v(\$0) = 0$$

For the gamble: $g = (.5 \circ \$100, .5 \circ \$0)$.

$$u(.5 \circ \$100, .5 \circ \$0) = .5$$

Let's say $u'(g) = \left(\sum p_i v(w_i)\right)^2$

$$u'(\$100) = v(\$100)^2 = 1$$

$$u'(\$0) = 0$$

$$u'(.5 \circ \$100, .5 \circ \$0) = .25$$

We have broken the expected utility property. The expected utility is still .5.

The expected utility property is only maintained by affine transformations to the utility function of the form:

$$u'(g) = \alpha u(g) + \beta$$

Expected outcomes and risk aversion.

Let's go back to a VNM utility function.

$$u(\$100) = 1$$

$$u(\$50) = .75$$

$$u(\$0) = 0$$

Consider the gamble $(.5 \circ \$100, .5 \circ \$0)$.

$$u(.5 \circ \$100, .5 \circ \$0) = .5$$

The expected *outcome* this gamble is $.5\,(100) + .5\,(0) = \$50$.

$$u\,(\$50) = .75$$

Notice that The utility of the expected wealth in the gamble is larger than the utility of the gamble itself. This is a demonstration of something we call "risk aversion"

$$E\,(g) \succ g$$

$$u\,(E\,(g)) > u\,(g)$$

$$u\left(\sum_{i=1}^{n} p_i w_i\right) > \sum_{i=1}^{n} p_i u\,(w_i)$$

Jensen Inequality:

By Jensen's inequality, $u\,(\sum_{i=1}^{n} p_i w_i) > \sum_{i=1}^{n} p_i u\,(w_i)$ is true if u is a concave function.

On the other hand, risk loving preferences:

$$E\,(g) \prec g$$

Are generated by *convex* utility over wealth. That is $u\,(w)$ is convex.

Certainty equivalents and risk premium.

A certainty equivalent CE is defined as the amount of sure wealth indifferent to a gamble. The CE of g is:

$$CE \sim g$$

Let's suppose the utility for sure amounts of wealth is: $v\,(w) = \sqrt{w}$

$$u\,(.5 \circ \$100, .5 \circ \$0) = 5$$

To find the CE, we need to find $\sqrt{CE} = 5$.

$$CE = 25$$

Risk Premium

We also call the difference between the $E(g)$ and CE the Risk Premium.

$$E(.5 \circ \$100, .5 \circ \$0) = \$50$$

$$CE = \$25$$

$$Risk\,Premium = \$25$$