Firm's Problem

Very general model of technology involves representing possibilities with vectors:

$$Y \subset \mathbb{R}^m$$

A vector like $(1, 1, -1)$ means "use one unit of good one and two and produce one unit of good three". This can be difficult to work with so we often just use a production function when there is only one good that is an output and the rest of goods are inputs:

$$f(\boldsymbol{x}) = y$$

The firm will be concerned with minimizing the cost of producing y

$$Min \sum_{i=1}^{n} w_i x_i \, S.t \, f(x) = y$$

Let's make some basic assumptions on f:

1. **Continuous**
2. **Strictly Increasing**
3. **Strictly Quasi-Concave**
4. $f(0) = 0$

Some Definitions:

Marginal Product of input i. This is analogous to marginal utility.

$$f_i(\boldsymbol{x}) = \frac{\partial (f(x))}{x_i}$$

Isoquants. This is analogous to indifference curve.

$$Q(y) = \{\boldsymbol{x} \geq 0 | f(\boldsymbol{x}) = y\}$$

MRTS: Marginal rate of technical substitution.

$$MRTS_{i,j} = -\frac{\frac{\partial(f(x))}{\partial x_i}}{\frac{\partial(f(x))}{\partial x_j}}$$

A Cost Minimization Problem:

$$Min \, w_1 x_1 + w_2 x_2 + w_3 x_3 - \lambda (x_1 x_2 x_3 - y)$$

First, does $x_1x_2x_3$ meet our assumptions? All are clear except strict quasi-concavity. The trick here is using the fact that a strictly monotonic transformation of a strictly concave function is strictly quasi-concave:

This function is strictly concave since it is the sum of strictly concave functions:

$$ln\left(x_1\right) + ln\left(x_2\right) + ln\left(x_3\right)$$

$e^{u(x)}$ is a strictly monotonic transformation. This gives us:

$$e^{ln(x_1)+ln(x_2)+ln(x_3)} = e^{ln(x_1)}e^{ln(x_2)}e^{ln(x_3)} = x_1x_2x_3$$

Let's move on to the first order conditions, knowing they will be sufficient for a cost minimization:

$$\frac{\partial\left(w_1x_1 + w_2x_2 + w_3x_3 - \lambda\left(x_1x_2x_3 - y\right)\right)}{\partial x_1} = w_1 - \lambda x_2x_3$$

$$\frac{\partial\left(w_1x_1 + w_2x_2 + w_3x_3 - \lambda\left(x_1x_2x_3 - y\right)\right)}{\partial x_2} = w_2 - \lambda x_1x_3$$

$$\frac{\partial\left(w_1x_1 + w_2x_2 + w_3x_3 - \lambda\left(x_1x_2x_3 - y\right)\right)}{\partial x_3} = w_3 - \lambda x_1x_2$$

$$\frac{x_2x_3}{w_1} = \frac{1}{\lambda}$$

$$\frac{x_1x_3}{w_2} = \frac{1}{\lambda}$$

$$\frac{x_1x_2}{w_3} = \frac{1}{\lambda}$$

$$x_1^3\frac{w_1^2}{w_2w_3} = y$$

$$x_1^* = y^{\frac{1}{3}}\left(\frac{w_1^2}{w_2w_3}\right)^{\frac{1}{3}}$$

$$x_2^* = y^{\frac{1}{3}}\left(\frac{w_3^2}{w_1w_3}\right)^{\frac{1}{3}}$$

$$x_3^* = y^{\frac{1}{3}}\left(\frac{w_3^2}{w_1w_2}\right)^{\frac{1}{3}}$$