Cobb Douglass Production Problem

$$x_1^\alpha x_2^\alpha$$

$$w_1 x_1 + w_2 x_2 + \lambda \left(y - x_1^\alpha x_2^\alpha\right)$$

$$\frac{\partial \left(w_1 x_1 + w_2 x_2 + \lambda \left(y - x_1^\alpha x_2^\alpha\right)\right)}{\partial x_1} = w_1 - \alpha \lambda x_1^{\alpha-1} x_2^\alpha$$

$$\frac{\partial \left(w_1 x_1 + w_2 x_2 + \lambda \left(y - x_1^\alpha x_2^\alpha\right)\right)}{\partial x_2} = w_2 - \alpha \lambda x_1^{\alpha} x_2^{\alpha-1}$$

$$\frac{w_1}{\alpha x_1^{\alpha-1} x_2^\alpha} = \lambda$$

The top is how much I save by decreasing input of x_1 by one unit.

Bottom is how much I decrease x_1 by to decrease output by one unit.

The ratio of these is how much cost savings I get by decreasing x_1 by enough to decrease output by one unit.

The FOC for input 2 gives us:

$$\frac{w_2}{\alpha x_2^{\alpha-1} x_1^\alpha} = \lambda$$

Combining these:

$$\frac{w_1}{\alpha x_1^{\alpha-1} x_2^\alpha} = \frac{w_2}{\alpha x_2^{\alpha-1} x_1^\alpha}$$

$$w_2 x_2 = w_1 x_1$$

$$x_2 = \frac{w_1}{w_2} x_1$$

Plug into production constraint: $x_1^\alpha x_2^\alpha = y$. This gives conditional input demand:

$$x_1 = \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}} y^{\frac{1}{2\alpha}}$$

$$x_2 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}} y^{\frac{1}{2\alpha}}$$

To get the cost function, plug these into $w_1 x_1 + w_2 x_2$ and then simplify:

$$c\left(y, w_1, w_2\right) = 2\left(w_1 w_2\right)^{\frac{1}{2}} y^{\frac{1}{2\alpha}}$$

Suppose $w_1 = w_2 = 1$ and $\alpha = \frac{1}{2}$

$$x_1 = y$$

$$x_2 = y$$

$$c = 2y$$

Homogeneous / Homothetic Cost and Input

Let's look at the cost function from the previous problem when $\alpha = \frac{1}{2}$:

$$c\left(1, w_1, w_2\right) = 2\left(w_1 w_2\right)^{\frac{1}{2}} y$$

When $\alpha = \frac{1}{2}$, the production function is homogeneous of degree 1- linearly homogeneous and this cost function is linear in y. In fact, it can be written:

$$c\left(1, w_1, w_2\right) y$$

This will be the form of the cost function for any homogeneous production function homogeneous of degree z.

$$c\left(1, w_1, w_2\right) y^{\frac{1}{z}}$$

We can construct the cost function knowing only the cost of producing one unit of output and the degree of homogeneity of the function.

This applies to input demands as well:

$$x_i\left(y, w_1, w_2\right) = x_i\left(1, w_1, w_2\right) y^{\frac{1}{z}}$$

Suppose a firm has homogeneous production of degree 1. They use one unit of input 1 and one unit of input 2 to produce one unit of output. This costs them \$10.

$$c\left(y, w_1, w_2\right) = 10y$$

$$x_1\left(y, w_1, w_2\right) = x_1\left(y, w_1, w_2\right) = y$$

There is some increasing $h\left(y\right)$ such that:

$$x = h\left(y\right) x\left(1, w_1, w_2\right)$$

$$c = h(y)\, c(1, w_1, w_2)$$

A Homothetic Production Problem

$$ln(x_1) + ln(x_2)$$

We know the first order condition from the Lagrangian will give this condition:

$$\frac{w_1}{\frac{\partial(ln(x_1)+ln(x_2))}{\partial x_1}} = \frac{w_2}{\frac{\partial(ln(x_1)+ln(x_2))}{\partial x_2}}$$

$$x_1 = \frac{w_2}{w_1} x_2$$

Plug this into the production function to find the conditional factor demands:

$$ln\left(\frac{w_2}{w_1} x_2\right) + ln(x_2) = y$$

Solving this gives:

$$x_2 = e^{\frac{y - ln\left(\frac{w_2}{w_1}\right)}{2}}$$

By analogy:

$$x_1 = e^{\frac{y - ln\left(\frac{w_1}{w_2}\right)}{2}}$$

Plugging this into the cost function:

$$c(y, w_1, w_2) = w_1 e^{\frac{y - ln\left(\frac{w_1}{w_2}\right)}{2}} + w_2 e^{\frac{y - ln\left(\frac{w_2}{w_1}\right)}{2}}$$

Do these have the form expected for a homothetic production function? It is not so easy to see, but the answer is *yes*:

$$x_1 = e^{\frac{y - ln\left(\frac{w_1}{w_2}\right)}{2}} = e^{\frac{y - 1 + 1 - ln\left(\frac{w_1}{w_2}\right)}{2}} = e^{\frac{y-1}{2}} e^{\frac{1 - ln\left(\frac{w_1}{w_2}\right)}{2}}$$

$$x_1(1, w_1, w_2) = e^{\frac{1 - ln\left(\frac{w_1}{w_2}\right)}{2}}$$

$$h(y) = e^{\frac{y-1}{2}}$$

Try at home to show that the cost function meets our expectations as well.

Separable Production:

$$ln\left(x_1\right)+ln\left(x_2\right)+x_3^{\frac{1}{2}}x_4^{\frac{1}{2}}$$

$$\frac{\frac{\partial\left(\log(x_1)+\log(x_2)+x_3^{\frac{1}{2}}x_4^{\frac{1}{2}}\right)}{\partial x_1}}{\frac{\partial\left(\log(x_1)+\log(x_2)+x_3^{\frac{1}{2}}x_4^{\frac{1}{2}}\right)}{\partial x_2}} = \frac{x_2}{x_1}$$

$$\frac{\frac{\partial\left(\log(x_1)+\log(x_2)+x_3^{\frac{1}{2}}x_4^{\frac{1}{2}}\right)}{\partial x_3}}{\frac{\partial\left(\log(x_1)+\log(x_2)+x_3^{\frac{1}{2}}x_4^{\frac{1}{2}}\right)}{\partial x_4}} = \frac{x_4}{x_3}$$

When we can partition the set of variables of a function into groups, such that the ratio of any two partial derivatives within the group does not depend on variables outside the group, we say that the function is separable on the groups.

The function above is separable on:

$$\left\{x_1,x_2\right\},\left\{x_3,x_4\right\}$$

We want to produce y_1 using x_1, x_2. What is the optimal amount of x_1 and x_2? Let's suppose $w_1 = w_2 = 1$. Using our work from previous lecture:

$$x_2 = e^{\frac{y_1}{2}}$$

$$x_1 = e^{\frac{y_1}{2}}$$

What is the cost of doing this?

$$c\left(\tilde{y},1,1\right) = 2e^{\frac{y_1}{2}}$$

Suppose I want to produce y_2 using x_3 and x_4. Borrowing our work from last lecture:

$$x_3 = y_2$$

$$x_4 = y_2$$

$$c\left(y_2,1,1\right) = 2y_2$$

We have found the cheapest way of producing y_1 from group $\{x_1, x_2\}$ and of producing y_2 from $\{x_3, x_4\}$. Thus, to produce $y = y_1 + y_2$ it will cost:

$$c(y_1 + y_2, 1, 1) = 2e^{\frac{y_1}{2}} + 2y_2$$

Minimizing this will provide the cheapest way of producing y.

$$Min_{y_1,y_2}\left(2e^{\frac{y_1}{2}} + 2y_2\right)$$

Since $y_1 + y_2 = y$. Let's plug in this constraint:

$$Min_{y_1}\left(2e^{\frac{y_1}{2}} + 2(y - y_1)\right)$$

$$\frac{\partial\left(\left(2e^{\frac{y_1}{2}} + 2(y - y_1)\right)\right)}{\partial y_1} = e^{\frac{y_1}{2}} - 2$$

First order condition:

$$e^{\frac{y_1}{2}} = 2$$

$$y_1 = 2\log(2)$$

$$y_2 = y - 2\log(2)$$

$$x_1 = x_2 = 2$$

$$x_3 = x_4 = y - 2\log(2)$$

Let's get some intuition for this solution:

$$f = ln(x_1) + ln(x_2) + x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}$$

Since $x_1 = x_2$ and $x_3 = x_4$ at the optimum we can write this as:

$$f = 2ln(x_1) + x_3$$

This is a quasi-linear function. The marginal product of x_1 is $\frac{2}{x_1}$ and marginal product of x_3 is 1. Since they cost the same, we should use x_1 (group 1) until it becomes less productive than x_3 (group 2) this happens at $x_1 = x_2 = 2$.

Profit Max:

Let's ignore the fact that price changes when output changes. We call this the "price taking" assumption. It is associated with a type of idealized market called "perfect competition". We will talk more about this assumption as we progress.

$$\pi = py - wx = pf(x) - wx$$

$$Max_x pf(x) - wx$$

The first order conditions:

$$pf_1(x) = w_1$$

$$pf_2(x) = w_2$$

$$\frac{w_1}{f_1(x)} = p$$

$$\frac{w_2}{f_2(x)} = p$$

$$\frac{w_1}{f_1(x)} = \frac{w_2}{f_2(x)}$$

FOCs for cost min:

$$\frac{w_1}{f_1(x)} = \lambda$$

$$\frac{w_2}{f_2(x)} = \lambda$$

$$\frac{w_1}{f_1(x)} = \frac{w_2}{f_2(x)}$$

Thus, cost minimization is implied by profit maximization. This implies we can write the profit function where c is the cost minimized cost function.

$$\pi^*(p, w_1, w_2) = Max_y [py - c(y, w_1, w_2)]$$

The Profit Function

$$\pi(p, w) = max_y py - c(y) = Max\, pf(x) - wx$$

When well defined:

1. Increasing in p,
2. Decreasing in w,
3. Homogeneous of degree one in p, w
4. Convex in p, w (why),
5. Hotelling:

$$\frac{\partial \pi}{\partial p} = y(p, w), -\frac{\partial \pi}{\partial w_i} = x_i(p, w)$$