Cournot Oligopoly

Suppose firms have the following technology:

$$f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$$

$$w_1 = w_2 = 1$$

The cost function of each firm is:

$$c(y_i) = 2y_i^2$$

Some notation. The total output chosen by everyone else is given by: $y_{-i} = \left(\sum_{i \in \{1,\ldots,J\}} y_i\right) - y_i = Y - y_i$.
Where $p_d(Y)$ is the inverse demand, profit is given by:

$$\pi(y_{-i}, y_i) = p_d(Y)\, y_i - c(y_i) = p_d(y_{-i}, y_i)\, y_i - c(y_i)$$

Let's assume $p_d(Y) = \frac{100}{Y}$

$$\pi(y_{-i}, y_i) = \frac{100}{Y} y_i - c(y_i) = \frac{100}{y_{-i} + y_i} y_i - 2y_i^2$$

This function is concave:

$$\frac{100}{y_{-i} + y_i} y_i - 2y_i^2$$

$-2y_i^2$ is concanve. So, if $\frac{100}{y_{-i}+y_i} y_i$ is concave in y_i then the whole function is concave because it is the sum of concave functions:

$$\frac{\partial \left(\frac{100}{y_{-i}+y_i} y_i\right)}{\partial y_i} = \frac{100}{y_{-i} + y_i} - \frac{100 y_i}{(y_{-i} + y_i)^2}$$

Revenue is strictly increasing when this derivative is positive. This is true when:

$$y_{-i} > 0$$

Revenue is strictly concave:

$$\frac{\partial \left(\frac{100}{y_{-i}+y_i} - \frac{100 y_i}{(y_{-i}+y_i)^2}\right)}{\partial y_i} < 0$$

$$0 < y_{-i}$$

So, the entire profit function is strictly concave if other firms are producing anything. Thus, the stationary point will give us the global maximum profit:

$$\frac{\partial \pi \left(y_{-i}, y_i\right)}{\partial y_i} = 0$$

$$\frac{\partial \left(\frac{100}{y_{-i}+y_i} y_i - 2y_i^2\right)}{\partial y_i} = 0$$

$$\frac{100}{y_{-i} + y_i} = \frac{100y_i}{\left(y_{-i} + y_i\right)^2} + 4y_i$$

Let's assume that the solution to this problem is symmetric. (There's no reason to think there isn't a symmetric solution). Thus, there is some y^* that solves this problem for every firm.

$$\frac{100}{\left(J-1\right) y^* + y^*} = \frac{100y^*}{\left(\left(J-1\right) y^* + y^*\right)^2} + 4y^*$$

$$\frac{100}{\left(J\right) y^*} = \frac{100y^*}{\left(\left(J\right) y^*\right)^2} + 4y^*$$

$$y^* = \frac{5\sqrt{J-1}}{J}$$

$$Y^* = 5\sqrt{J-1}$$

$$p^* = \frac{20}{\sqrt{J-1}}$$

Let's compare this to perfect competition. In a previous lecture we found in perfect competition:

$$y^* = \frac{5\sqrt{J}}{J} = \frac{5}{\sqrt{J}}$$

$$Y^* = 5\sqrt{J}$$

$$p^* = \frac{20}{\sqrt{J}}$$

Let's make some tables:

$$Round[Table[\{J, \frac{20}{\sqrt{J-1}}, \frac{20}{\sqrt{J}}\}, \{J, \{2, 10, 100, 1000\}\}], 0.001]$$

Equilibrium price:

$$\begin{pmatrix} 2. & 20. & 14.142 \\ 10. & 6.667 & 6.325 \\ 100. & 2.01 & 2. \\ 1000. & 0.633 & 0.632 \end{pmatrix}$$

Equilibrium Supply:

$$\begin{pmatrix} 2. & 5. & 7.071 \\ 10. & 15. & 15.811 \\ 100. & 49.749 & 50. \\ 1000. & 158.035 & 158.114 \end{pmatrix}$$

When to impose symmetry:

A caveat. The symmetry condition needs to be imposed after the derivative has been taken. Otherwise, the firm is not solving for what is optimal for them conditional on a fixed amount of output of the other firms. They are treating their profit function as if they can change every firm's output at the same time. That is not in their control. Compare these marginal revenues:

$$\frac{\partial \left(p\left(y_i + y_{-i}\right) y\right)}{\partial y} = p'\left(Y\right) y + p'\left(Y\right)$$

$$\frac{\partial p\left(Jy\right) y}{\partial y} = Jp'\left(Y\right) y + p'\left(Y\right)$$

The indirect effect of output on revenue is J times higher if the firm can change everyone's output. In fact, what they end up doing is acting as a monopolist if symmetry is imposed before taking the derivative.

Cournot Oligopoly- Different Costs

Let's now consider two firms with different cost structure.

$$c_1\left(y_1\right) = 2y_1^2$$

$$c_2\left(y_2\right)=2y_2^{\frac{3}{2}}$$

$$y_d=\frac{100}{y}$$

$$\pi_1\left(y_1\right)=\frac{100}{y_1+y_2}y_1-2y_1^2$$

$$\pi_2\left(y_2\right)=\frac{100}{y_1+y_2}y_2-2y_2^{\frac{3}{2}}$$

$$\frac{\partial\left(\frac{100}{y_1+y_2}y_1-2y_1^2\right)}{\partial y_1} = -\frac{100y_1}{\left(y_1+y_2\right)^2}-4y_1+\frac{100}{y_1+y_2}$$

$$\frac{\partial\left(\frac{100}{y_1+y_2}y_2-2y_2^{\frac{3}{2}}\right)}{\partial y_2} = -\frac{100y_2}{\left(y_1+y_2\right)^2}-3\sqrt{y_2}+\frac{100}{y_1+y_2}$$

FOC for firm 1

$$-\frac{100y_1}{\left(y_1+y_2\right)^2}-4y_1+\frac{100}{y_1+y_2}==0$$

$$y_1=\frac{1}{3}\left(-\frac{\sqrt[3]{2}y_2^2}{\sqrt[3]{-2y_2^3-675y_2+15\sqrt{3}\sqrt{4y_2^4+675y_2^2}}}-2y_2-\frac{\sqrt[3]{-2y_2^3-675y_2+15\sqrt{3}\sqrt{4y_2^4+675y_2^2}}}{\sqrt[3]{2}}\right)$$

FOC for firm 2

$$-\frac{100y_2}{\left(y_1+y_2\right)^2}-3\sqrt{y_2}+\frac{100}{y_1+y_2}==0$$

Let's find a solution:

$$y_1=2.42755, y_2=3.95231$$

In cournot with same technology as firm one, two firms each produce:

$$y^*=\frac{5\sqrt{2-1}}{2}=2.5$$

Stakleberg Oligopoly.

Now, let's assume that firms don't move at the same time. We need to change the demand function to simplify things a bit since the model is getting more complicated.

$$p_d = 100 - y$$

Both firms have cost function:

$$c(y_i) = 2y_i^2$$

Firm 2 observes firm 1's quantity before choosing their quantity. Firm 2 chooses to maximize their profit:

$$\pi_2(y_1, y_2) = (100 - (y_1 + y_2))\, y_2 - 2y_2^2$$

$$\frac{\partial\left((100 - (y_1 + y_2))\, y_2 - 2y_2^2\right)}{\partial y_2} = -y_1 - 6y_2 + 100$$

Best response of firm 2:

$$y_2 = \frac{100 - y_1}{6}$$

If the firms did more in sequence, they both would have this best response and the symmetric equilibrium is:

$$y^* = 14.2857$$

What's different is, firm 1 know's firm 2 will behave this way.

$$\pi_1(y_1, y_2(y_1)) = \left(100 - y_1 - \frac{100 - y_1}{6}\right) y_1 - 2y_1^2$$

$$\frac{\partial\left(\left(100 - y_1 - \frac{100-y_1}{6}\right) y_1 - 2y_1^2\right)}{\partial y_1} = \frac{1}{6}(y_1 - 100) - \frac{35y_1}{6} + 100$$

The optimal output for firm one is:

$$y_1 = \frac{250}{17.0} = 14.7059$$

This implies firm two produces:

$$y_2 = 14.2157$$