The Slutsky Equation

Start with Duality:

$$x_i^h(p,u) = x_i(p, e(p,u))$$

Now take the total derivative and move things around:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - \frac{\partial x_i}{\partial y}\frac{\partial e}{\partial p_i}$$

Examples- Cobb-Douglas

Suppose $u = x_1 x_2$

Demands:

$$L = x_1 x_2 - \lambda(x_1 + x_2 - m)$$

$$\frac{\partial(x_1 x_2 - \lambda(p_1 x_1 + p_2 x_2 - m))}{\partial x_1} = x_2 - \lambda p_1$$

$$\frac{\partial(x_1 x_2 - \lambda(p_1 x_1 + p_2 x_2 - m))}{\partial x_2} = x_1 - \lambda p_2$$

$$\frac{x_2}{p_1} = \lambda$$

$$\frac{x_1}{p_2} = \lambda$$

Notice, the Lagrange Multiplier is exactly equal to the amount utility increases when I spend marginally more on either good. Setting these equal,

$$x_1 = x_2$$

Plugging this into the budget equation: $x_1 + x_2 = m$

$$x_1 = x_2 = \frac{1}{2}\frac{m}{p_i}$$

Indirect Utility:

Plug the Marshallian demands into the utility function:

$$u = \frac{1}{2}\frac{m}{p_1}\frac{1}{2}\frac{m}{p_2}$$

$$v\left(p_1, p_2, m\right) = \frac{1}{4}\frac{m^2}{p_1 p_2}$$

Expenditure:
Invert the indirect utility function, solving for m:

$$e = \sqrt{4up_1p_2}$$

Hicksian Demands:
Leverage the envelope condition, take derivative of the expenditure function to get Hicksian demands:

$$\frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_1} = x_1^h$$

$$x_1^h = \sqrt{u}\sqrt{\frac{p_2}{p_1}}$$

$$x_2^h = \sqrt{u}\sqrt{\frac{p_1}{p_2}}$$

Let's decompose demand:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - \frac{\partial x_i}{\partial m}\frac{\partial e}{\partial p_i}$$

Plug these in, take the derivatives and clean up the result:

$$-\frac{m}{2p_1^2} = -\frac{1}{2}\frac{\sqrt{p_2}\sqrt{u}}{p_1^{1.5}} - \frac{1}{2}\frac{\sqrt{p_2}\sqrt{u}}{p_1^{1.5}}$$

Notice that we have an m on the right and utilities on the left. To show these are equal, replace income with expenditure through duality:

$$-\frac{\sqrt{up_2}}{p_1^{1.5}} = -\frac{\sqrt{up_2}}{p_1^{1.5}}$$

We have shown the Slutsky equation holds for this consumer and that substitution and income effect are the same. Notice that the substitution effect and income effect are both:

$$-\frac{1}{2}\frac{\sqrt{p_2}\sqrt{u}}{p_1^{1.5}} = -\frac{1}{2}\frac{\sqrt{p_2}\sqrt{u}}{p_1^{1.5}}$$

Thus, we have shown that for a Cobb-Douglas consumer with utility of the form:

$$x_1^{\alpha} x_2^{\alpha}$$

Substitution and income effect are always equal.