More Oligopoly

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1 CES and Fermat's Last Theorem

Suppose we have this CES funtion:

$$\left(x_1^{-n} + x_2^{-n}\right)^{-\frac{1}{n}} = y$$

For $n = 1$, this is:

$$\left(x_1^{-1} + x_2^{-1}\right) = y^{-1}$$

Let's look for instances where the firms use an integer amount of both inputs and end up with an integer amount of output. That would be interger solutions to this:

$$\left(x_1^{-n} + x_2^{-n}\right) = y^{-n}$$

This is the same as (multiplying both sides by $(x_1 x_2 y)^n$:

$$\left((x_2 y)^n + (x_1 y)^n\right) = (x_1 x_2)^n$$

This implies if we had an integer solution to the equation with power $-n$, we would have one for n as well. That is, an interger solution to an equation like:

$$x^n + y^n = z^n$$

By Fermat's Last theorem, there are no (non-trivial) integer solutions (or even rational ones) to this equation for interger $n > 2$.

2 Two Firm Cournot, Generic Productivity

Firms with production function $x_1^{\alpha} x_2^{\beta}$ with x_2 fixed at 1 and inverse demand: $p(Q) = \frac{294}{Q}$. The choose q_i simultaneously. As before, $w_1 = 4$ and $w_2 = 1$.

Let's find the cost function. The conditional factor demand in the short-run is:

$$x_1 = q^{\frac{1}{\alpha}}$$

This implies the cost function is:

$$c\left(q\right) = 4q^{\frac{1}{\alpha}} + 1$$

This gives the profit function:

$$\pi_1\left(q_1, q_2\right) = \frac{294}{q_1 + q_2} q_1 - \left(4q_1^{\frac{1}{\alpha}} + 1\right)$$

The marginal profit is:

$$\frac{\partial\left(\frac{294}{q_1+q_2} q_1 - \left(4q_1^{\frac{1}{\alpha}} + 1\right)\right)}{\partial q_1} = -\frac{4q_1^{\frac{1}{\alpha}-1}}{\alpha} - \frac{294 q_1}{\left(q_1 + q_2\right)^2} + \frac{294}{q_1 + q_2}$$

First-order condition:

$$-\frac{4q_1^{\frac{1}{\alpha}-1}}{\alpha} - \frac{294 q_1}{\left(q_1 + q_2\right)^2} + \frac{294}{q_1 + q_2} = 0$$

Since the firms are symmetric, let's look for a symmetric solution setting $q = q_1 = q_2$

$$-\frac{4q^{\frac{1}{\alpha}-1}}{\alpha} - \frac{294q}{\left(2q\right)^2} + \frac{294}{2q} = 0$$

$$q = \left(\frac{147}{8}\right)^{\alpha}\left(\frac{1}{\alpha}\right)^{-\alpha}$$

Suppose you observe q and want to solve for α :

$$q = \left(\frac{147}{8}\alpha\right)^{\alpha}$$

This is a curious function to appear here, it is of the form $y = x^x$. Lambert-W shows up again:

$$\alpha = \frac{\log(q)}{W\left(\frac{147\log(q)}{8}\right)}$$

3 Two Firm Cournot- Different Technology

Let's now consider two firms with different cost structure and inverse demand: $p(Q) = \frac{294}{Q}$.

$$c_1(q_1) = 4q_1^2 + 1$$

$$c_2(q_2) = 4q_2^{\frac{3}{2}} + 1$$

Firm 1 has the same profit function as before.

$$\pi_1(q_1, q_2) = \frac{294}{q_1 + q_2} q_1 - \left(4q_1^2 + 1\right)$$

Firm 2's is different:

$$\pi_2(q_1, q_2) = \frac{294}{q_1 + q_2} q_2 - \left(4q_2^{\frac{3}{2}} + 1\right)$$

The first-order conditions:

$$\frac{\partial\left(\frac{294}{q_1+q_2} q_1 - \left(4q_1^2 + 1\right)\right)}{\partial q_1} = -\frac{294q_1}{(q_1 + q_2)^2} - 8q_1 + \frac{294}{q_1 + q_2} = 0$$

$$\frac{\partial\left(\frac{294}{q_1+q_2} q_2 - \left(4q_2^{\frac{3}{2}} + 1\right)\right)}{\partial q_2} = -\frac{294q_2}{(q_1 + q_2)^2} - 6\sqrt{q_2} + \frac{294}{q_1 + q_2} = 0$$

This is a rather complex set of equations that needs to be solved simultaneously. A symbolic manipulation might produce some nasty results. Let's just ask mathematica to give us an approximate numerical solution:

$$NSolve[-\frac{294q_1}{(q_1 + q_2)^2} - 8q_1 + \frac{294}{q_1 + q_2} == 0, -\frac{294q_2}{(q_1 + q_2)^2} - 6\sqrt{q_2} + \frac{294}{q_1 + q_2} == 0, \{q_1, q_2\}]$$

$$q_2 \approx 5.06, q_1 \approx 2.92$$

Nice work mathematica!

4 Stackelberg: Linear Demand

Two firm's have cost functions: $4q_i + 1$ and demand is now the simpler linear form: $294 - (q_1 + q_2)$. Firm one chooses a quantity, then firm two observes this and chooses a quantity.

Firm 2 observes firm 1 output and solves:

$$\frac{\partial \left(\left(294 - \left(q_1 + q_2 \right) \right) q_2 - \left(4q_2 + 1 \right) \right)}{\partial q_2} \quad = \quad -q_1 - 2q_2 + 290 = 0$$

This gives us Firm 2's reaction function:

$$\frac{290 - q_1}{2} = q_2$$

Firm 1 knows that firm 2 behaves optimally and can also calculate this reaction function. Note that in solving for it's own quantity, firm one cannot assume that firm two's quantity is fixed. This is nonsense. Firm two reacts to firm one and so firm one must account for this reaction in it's optimization. This is it's first-order condition:

$$\frac{\partial \left(\left(294 - \left(q_1 + \frac{290 - q_1}{2} \right) \right) q_1 - \left(4q_1 + 1 \right) \right)}{\partial q_1} = 0$$

$$\frac{1}{2} \left(q_1 - 290 \right) - \frac{3q_1}{2} + 290 = 0$$

$$q_1 = 145$$

$$q_2 = \frac{145.0}{2} = 72.5$$

Let's compare this to if firms moves sumultaneously, both would solve:

$$\frac{\partial \left(\left(294 - \left(q_1 + q_2 \right) \right) q_2 - \left(4q_2 + 1 \right) \right)}{\partial q_2} = -q_1 - 2q_2 + 290 = 0$$

Both have the same first-order condition. We can solve for a symmetric equilibrium:

$$q = 96.6667$$

Notice that, by moving first, firm one ends up with higher profit than wehn they move simultaneously. There is a first mover advantage. Is there always a first mover advantage? No. In fact, if we use the non-lienar demand function we have been looking at elsewhere recently, there is no first-mover advantage.

5 Stackelberg: Non-Linear Demand

Two firm's have cost functions: $4q_2 + 1$ and demand is now: $\frac{294}{q_1+q_2}$. Firm one chooses a quantity, then firm two observes this and chooses a quantity. Firm two's first order condition is:

$$\frac{\partial\left(\frac{294}{q_1+q_2}q_2 - (4q_2+1)\right)}{\partial q_2} = -\frac{294q_2}{(q_1+q_2)^2} + \frac{294}{q_1+q_2} - 4 = 0$$

This has two solutions, but only one positive solution:

$$q_2 = \frac{1}{2}\left(7\sqrt{6}\sqrt{q_1} - 2q_1\right)$$

Note that unlike with linear demand, this reaction function is non-linear in q_1. Let's have a look at it's slope:

$$\frac{\partial\left(\frac{1}{2}\left(7\sqrt{6}\sqrt{q_1} - 2q_1\right)\right)}{\partial q_1} = \frac{1}{2}\left(\frac{7\sqrt{\frac{3}{2}}}{\sqrt{q_1}} - 2\right)$$

q_2 is decreasing in q_1 when:

$$\frac{147}{8} < q_1$$

Thus, q_2 is stationary in q_1 when $q_1 = \frac{147}{8}$. This is precisely the q that firms use in cournot oligopoly. Thus, the cournot oligopoly level of quantity: $\frac{147}{8}$ also solves this function:

$$\frac{\partial\left(\frac{294}{q_1+q_2(q_1)}q_1 - (4q_1+1)\right)}{\partial q_1} = 0$$

Note that this differs from the cournot first-order condition for firm 1 in the inclusion of the reaction function $q_2(q_1)$. However, at $q_1 = \frac{147}{8}$, this funciton is stationary in q_1. Thus, the entire profit funciton is also stationary at $q_1 = \frac{147}{8}$ and solves firm 1's first-mover profit maximiation problem. We already know that $q_2 = \frac{147}{8}$ is optimal when firm one sets $q_1 = \frac{147}{8}$ because firm 2's first order condition is the same as in the simultaneous-move cournot model. Because of the particular special form of demand, it turns out that the simultaneous-move and the sequential-move models result in the exact same outcome in this case. There is no first-mover advantage.