Matching Soulmates*

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Abstract

We study iterated matching of soulmates [IMS], a process of forming coalitions that are the favorite for each member (soulmates), coalitions of soulmates in the remaining group, and so on. If all players can be matched as soulmates, we say that preferences are IMS-complete. It is known that under some sets of assumptions implying IMS completeness, desirable stability and incentive properties hold. We allow environments with IMS-incomplete preferences and demonstrate desirable properties continue to hold for those players who are matched as soulmates. One such property is that coalitions produced by IMS belong to any stable partition and mechanisms that implement IMS give players in these coalitions no incentive to deviate from truthful preference reporting. Another is that players who believe that they will be matched as soulmates in any outcome that has positive probability have no incentive to deviate from truth telling, even as members of coalitions. Using real-world data and simulation we demonstrate that scenarios in which many people are matched by IMS are common under natural classes of preferences.

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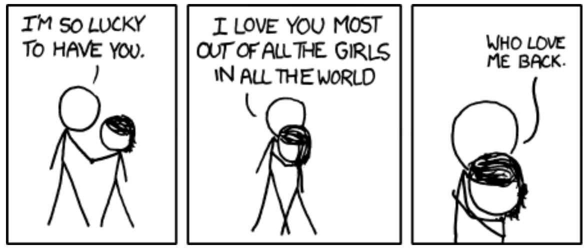

1 Introduction

We study iterated matching of soulmates [IMS], an algorithm for forming coalitions that are the favorite for each of their members (soulmates), coalitions of soulmates in the remaining group, and so on. If all players can thereby be matched as soulmates, we say that their preferences are *IMS-complete*. The IMS algorithm has been studied in a number of papers under conditions ensuring IMS-completeness. We note especially Banerjee, Konishi and Sönmez (2001), which demonstrates that, given preferences, outcomes of the algorithm are in the core, and Pápai (2004), which demonstrates that to ensure IMS-completeness for any set of preferences, the domain of preferences must be severely restricted. In many real world situations, however, it may well be that matching soulmates is highly desirable, even when preferences are *IMS-incomplete* (that is, fail to be IMS-complete); we allow such situations. Our paper provides the first study of IMS as a general design criterion and presents its remarkable consequences for stability and incentives.

Let us begin by illustrating an environment in which preferences are IMS-complete and our results simply extend those in the literature to a larger domain. Then we modify preferences so that IMS completeness does not hold but our results continue to apply to those players matched as soulmates.

An example with IMS-complete preferences. Suppose Alice would rather be with Alex than anyone else. Alex feels the same way about Alice. They are *soulmates*. Since they would be willing to leave any other partners to be together, they are a threat to the stability of *all* matchings that do not pair them together.

Bertie and Ben are not so smitten with each other; Ben would rather be with Alice than anyone, and Bertie with Alex. But Ben and Bertie know Alice and Alex are soulmates. As long as Alice and Alex are paired, Bertie and Ben are soulmates in the remaining set of players. With Alice and Alex paired, Bertie and Ben would block the stability of any matching in which they are not paired.

Kira and Casey are not deeply smitten with each other. But, if they are the only two left, then in any stable match, since Alice must be matched with Alex and Bertie must be matched with Ben, Kira must be matched with Casey unless one of them prefers to remain alone. Thus, in this six-player matching market, Alice with Alex, Bertie with Ben, Kira with Casey is the unique stable match.

It was found simply by iteratively forming soulmate coalitions; eventually all players were matched as soulmates.

The model is not limited to two-sided matching. It extends to matching environments more generally – even where coalitions are not limited to two players. Suppose our six players can profitably form coalitions of up to three players. Suppose Kira, Alice and Alex work well together and would each prefer this coalition to any other. Bertie and Ben would most like to be matched together along with a third partner with an "A" name, but they disagree on whether Alice or Alex would be best as the third partner. Further, they would rather be paired just with each other than include someone who is not in the "A" family. With these preferences, Kira, Alice and Alex are soulmates. They would block any collection of coalitions that does not include them. Thus, Bertie and Ben cannot have a third partner from the "A" family and are conditional soulmates; once Kira, Alice and Alex are matched, Bertie and Ben are each other's soulmates. Finally, this leaves Casey alone. The coalition structure created by IMS, specifically, $\{\{Kira, Alice, Alex\}, \{Bertie, Ben\}, \{Casey\}\}$ is the unique core coalition structure.

Further, it continues to hold that no set of players can benefit by jointly misstating their preferences. Kira, Alice and Alex already have their favorite coalition, so no group of players including any of these three can simultaneously benefit by joint misstatement of their preferences. Bertie and Ben could do better only with the addition of Alice or Alex to their coalition. But, for that to happen, Bertie and Ben would have to convince Alice or Alex to misstate their preferences, which they will not do. The only person left is Casey but, to benefit from a misstatement, Casey would have to get one of the other five to misstate their preferences, which they have no incentive to do.

Under the happy circumstance that a profile is IMS-complete, the match resulting from IMS is stable and no player can benefit from misstating their preferences- truth telling is a strong Nash equilibrium in any mechanism that implements IMS. The above stability results are simply extensions of results already in Banerjee, Konishi and Sönmez (2001) for environments satisfying the "top coalition" property introduced there to environments satisfying IMS completeness; see *Supplementary Material* for a comprehensive discussion comparing IMS-completeness with other related conditions in the literature.

Examples with IMS-incomplete preferences. It may be that preferences among the four players Bertie, Ben, Casey and Kira are cyclic; Bertie may most prefer to be with Ben, who would most prefer to be with Casey, who would most prefer to be with Kira who prefers Bertie. Then IMS would match Alice and Alex, but not any of the other players. Still, in any stable match, as we demonstrate more generally below, Alex and Alice must be paired and they have no incentive to deviate from truth-telling. In general, in the coalitional game restricted to the set of players who are soulmates, the core partition of players is nonempty and contains all soulmate coalitions.

In addition, even when preferences may be IMS-incomplete, a player *may* *believe* that when she reports her preferences truthfully, she will be matched as a soulmate in any outcome with positive probability. In this case, truthful revelation is, for her, a best response. Moreover, when she reports her preferences truthfully, she is not a member of any jointly-profitable deviation from truth-telling.

These new results appear to have wide applicability. For instance, suppose Alice is participating in a job matching mechanism that implements IMS. She believes that all the companies will report the same strict preferences over her and the other candidates, but she does not know what those preferences are. Under these beliefs, she will surely become a soulmate once the higher ranking candidates have been matched with their soulmates. In this scenario, Alice has no incentive to deviate from truthful reporting, even as a member of a coalition.

Our main contributions. As illustrated by our examples above, even with IMS-*in*complete profiles, IMS has desirable stability properties for those players who are matched under IMS. Similarly, IMS can be used as a preprocessing step for any mechanism and with *any* domain of preferences.[1] In any of these cases, preprocessing a mechanism with IMS guarantees valuable stability and incentive properties for the players who are soulmates

IMS can also be invoked as a mechanism design criterion. In special circumstances, players that are matched by IMS — soulmates — are also matched under some familiar mechanisms, such as Deferred Acceptance (see Gale and Shapley, 1962) and Top Trading Cycles (Shapley and Scarf, 1974). In contrast, other mechanisms, such as Random Serial Dictatorship (Abdulkadiroğlu and Sönmez, 1998) and the Boston School Choice mechanisms (Abdulkadiroğlu and Sönmez, 2003), may not match soulmates.

As indicated earlier, as an algorithm IMS has appeared on several occasions in the literature (e.g. Banerjee, Konishi and Sönmez, 2001; Bogomolnaia and Jackson, 2002; Pápai, 2004). So far, however, IMS has only been used either (i) as a way to prove the existence of a stable match for a class of preference profiles, or (ii) as a mechanism on domains containing only *IMS-complete* profiles. Banerjee, Konishi and Sönmez (2001) and Pápai (2004) use conditions implying IMS-completeness to guarantee non-emptiness of the core.

Part of the motivation of our paper is to understand what can be done towards achieving "second best" properties through IMS even in situations where desiderata such as truthful Nash equilibria or stability cannot be fully satisfied. In this sense, our paper is related to a recent strand of literature that studies the partial satisfaction of non-manipulability and stability conditions (see, e.g., Parkes, Kalgnanam and Eso, 2002; Pathak and Sönmez, 2013; Mennle and Seuken, 2014; Andersson, Ehlers and Svensson, 2014; Arribillaga and Massó, 2015; Decerf and Van der Linden, 2016*a*,*b*; Chen and Kesten, 2017)

[1] By "preprocessing a mechanism with IMS", we mean first matching the players who match under IMS, and then applying the mechanism to the remaining set of players.

IMS empirically and computationally. Our empirical analysis studies three settings: (i) a roommates problem[2] using data from a university social network; (ii) a similar problem involving building work teams using data on connections within a consulting firm; and (iii) a two-sided matching problem using data from a speed-dating experiment. A surprising number of people can be matched by IMS; about 25% to 40% on average in the three environments and as many as 75% for particular instances of work-teams.[3]

Our computational experiments in Supplementary Material, Section C.1 analyze how common IMS-complete profiles are in the roommates environment and how likely players are to be matched by IMS. We consider unconstrained preference profiles as well as profiles that are a relaxation of reciprocal preferences,[4] and profiles that are a relaxation of common-ranking. While IMS-complete preferences are rare among unconstrained preferences, they are quite common when preferences exhibit strong reciprocity or are close to commonly ranked. Section 6 contains the results of our computational and empirical analysis of soulmates.

The structure of our paper The structure of our paper is as follows. In Section 2 we present the general hedonic matching environment as well as several definitions used throughout the paper. In Section 3 we define IMS formally and provide several examples. We turn to our key results on incentive compatibility and stability in Sections 4 and 5 respectively.

In Supplementary Material we also study the commonality of soulmates and conditional soulmates in both computational experiments and in empirical analysis of three real-world datasets. As Roth (2002) writes: "in the service of design, experimental and computational economics are natural complements to game theory," as they are for us. The Supplementary Material also contains analysis of the relationship of IMS to other conditions in the literature.

2 Environment

We closely follow the model of Banerjee, Konishi and Sönmez (2001). However, unlike Banerjee, Konishi and Sönmez (2001), but as in Pápai (2004) we will be concerned with situations where players *report* their preferences to a mechanism that then forms coalitions based on these reported preferences (as opposed to an environment in which players' preferences are *known* to the mechanism designer).

[2]The roommates problem was originally introduced by Gale and Shapley (1962) as an extension of the marriage and matching problems (Gale and Shapley, 1962; Roth, 2002; Bojilov and Galichon, 2016). The problem considers pairing players into roommates or partners (Kaya and Vereshchagina, 2015). For a recent review of the literature related to the problem, see Manlove (2013).

[3]The proportions depend on how indifferences are broken. See Leo et al. (2016) for results on non-strict preferences and IMS.

[4]We consider profiles where player i's top k partners also list i among their top k partners. We refer to these as k-reciprocal profiles.

The **total player set** is given by $N = \{1, \ldots, n\}$. A **coalition** is a nonempty set of players $C \in 2^N \backslash \emptyset$. Each player $i \in N$ has a complete and transitive preference $\succ_i$ over the collection of coalitions to which she may belong, denoted C_i for player i. Equivalently, we can view $\succ_i$ as a preference relation over coalitions of players that would join i (that is, it represents i's preferences over potential teammates). Coalition formation problems with this feature are known as "hedonic" (Drèze and Greenberg, 1980). As the notation $\succ_i$ suggests, we assume that preferences over coalitions are *strict*.[5] The domain of i's possible preferences is D_i. A coalition $C \in C_i$ is **acceptable** for i if $C = \{i\}$ or $C \succ_i \{i\}$.

A **profile** (of preferences) $\succ \in D = \times_{i \in N} D_i$ is a list of preferences, one for each player in N. Given profile $\succ \in D$ and any subset $S \subseteq N$, the **subprofile** (of preferences) for players in S is denoted by $\succ_S \in D_S = \times_{i \in S} D_i$. As is customary, we let $\succ_{-i} = \succ_{N \backslash \{i\}}$.

The domain of possible ***true*** **profiles** is $R \subseteq D$. In general, the domain of true profiles R need not to be equal to D. Although a preference $\succ_i \in D_i$ may be a "conceivable" preference for i, $\succ_i$ need not be player i's preference in any *true* profile $\succ \in R$.[6] It is also possible that, although all preferences $\succ_i \in D_i$ are i's true preference for *some* profile $\succ \in R$, some profiles $(\succ_i, \succ'_{-i})$ with $\succ'_{-i} \neq \succ_{-i}$ are not elements of R because true preferences are *interdependent*. (That is, $\succ'_{-i}$ cannot be the subprofile for players in $N \backslash \{i\}$ when i's preference is $\succ_i$. See, for example, the domains of *k-reciprocal* profiles in Section C.2). Domain R is **Cartesian** if $R = \times_{i \in N} R_i$ for some collection of individual domains R_i.

A **coalition structure** π is a partition of N. For any coalition structure π and any player $i \in N$, let π_i denote i's coalition of membership, that is, the coalition in π which contains i.

A **(direct coalition formation) mechanism** is a game form M that associates every *reported* profile $\succ' \in D$ with a coalition structure $\pi \in \Pi$ (Π is the set of all coalition structures). For every $i \in N$, the set of preferences D_i is i's strategy space for the mechanism M. Because mechanisms are *simultaneous* game forms, D must be the Cartesian product of the sets D_i; simultaneity makes it impossible for any player or group of players to condition their reports on the report of other players.[7]

Together, a pair $(M, \succ)$, where $\succ \in R$ is a profile of true preferences, determines a **preference revelation game**. Again, the strategy space of a player i in this game is the set of her reported preferences D_i. Once profile $\succ' \in D$ is reported, the mechanism M determines a coalition structure, denoted $M(\succ')$. The coalition containing player i is denoted $M_i(\succ')$; we say that M **matches** i with $M_i(\succ')$. Each player i evaluates their assigned coalition according to their *true* preference $\succ_i$.

[5] In an associated working paper (Leo et al., 2016), we study extensions of the results in this paper allowing for the possibility of indifferences.

[6] For example, a preference $\succ_i$ in which only $\{i\}$ is acceptable for i is conceivable. However, the mechanism designer may believe that i's domain of true preferences R contains preferences in which at least one pair $\{i, j\}$ is acceptable for i.

[7] Such conditioning would require M to be a sequential mechanism, which is *not* allowed in this paper. Recall that, unlike the space of strategy profiles D, the domain of *true* profiles R needs *not* be Cartesian (see above).

This model of a direct coalition formation mechanisms generalizes many common matching environments. When mechanisms are individually rational, restrictions on the collection of feasible coalitions can often be translated into restrictions on the domain of preferences by forcing infeasible coalitions that contain i to be "*un*acceptable" for i (i.e., ordered strictly below i given $\succ_i$). As an example, in our environment, the roommates problem is obtained by restricting D to the set of profiles in which only singletons and pairs of players are acceptable. In the marriage problem (Gale and Shapley, 1962), only singletons and pairs of players with a player from each side of the market are acceptable. In the college admission problem (Roth, 1985), D is such that only coalitions containing a single college and some (or no) students are acceptable (and that students are indifferent between any two coalitions with the same college).

2.1 Incentives and Outcome Properties of Mechanisms

In this section, we introduce two kinds of desirable properties of mechanisms. The first kind are properties that capture the incentives for players to report preferences truthfully. The second kind are properties of the *outcomes* of the mechanism with respect to the *reported* preferences.

There are at least two reasons to favor mechanisms that provide incentives to report preferences truthfully. First, if players do not report preferences truthfully, then desirable properties that the mechanism satisfies with respect to the *reported* preferences might not be satisfied with respect to the *true* preferences.[8] Second, mechanisms with good incentives to report preferences truthfully "level the playing field" (Pathak and Sönmez, 2008) by protecting naive players who report preferences truthfully against the manipulations of more strategically skilled players.[9]

When players have incentives to report preferences truthfully, properties with respect to *reported* preferences are good approximations of properties with respect to true preferences. Properties with respect to reported preferences may also be relevant *per se* to the mechanism designer. For example, in school choice, a mechanism that selects a *core* matching with respect to the *reported* preferences provides a protection against challenges of the matching in court.[10]

We first introduce incentive properties. Given a true profile $\succ \in R$, game $(M, \succ)$ has a **truthful strong Nash equilibrium** if there exists no group of

[8] As illustrated at the end of Example 5, a mechanism can, for example, produce a *core* outcome with respect to the reported preferences that is not even Pareto optimal with respect to the true preferences.

[9] For empirical evidence on the loss incurred by naive players in mechanisms with low incentives to be truthful, see the school choice laboratory experiments in Basteck and Mantovani (2016*a*) and Basteck and Mantovani (2016*b*). See also Pathak and Sönmez (2008) for a theoretical argument in the case of school choice.

[10] If the selected matching is not a core matching, it could be challenged in courts on the basis that students' priorities at schools have not been respected. It seems plausible that courts will rule based on reported preferences rather than true preferences. It is harder to imagine a court ruling in favor of a student who complains about a mechanism's outcome based on unreported true preferences.

players $S \subseteq N$ and no reported subprofile $\succ'_S \in D_S$ different from $\succ_S$ such that

$$M_i(\succ'_S, \succ_{N\setminus S}) \succ_i M_i(\succ_S, \succ_{N\setminus S}) \qquad \text{for all } i \in S.^{11} \tag{1}$$

Often, we care about whether a mechanism M induces games that have a truthful strong Nash equilibrium for *every* profile in the domain of true profiles R. Mechanism M is said to have a **truthful strong Nash equilibrium on domain** R if $(M, \succ)$ has a truthful strong Nash equilibrium for all $\succ \in R$.

We now introduce properties of the *outcomes* of a mechanism, where these properties are evaluated with respect to the *reported* preferences. We focus on the core and the properties of blocking coalitions.

Given any profile $\succ \in D$, a core partition is a coalition structure π^* in which no subset of players strictly prefers matching with each other rather than matching with their respective coalitions in π^*. Formally, π^* is a **core partition** if there does not exist a **blocking coalition** to π^*, that is, a coalition $C \in 2^N$ such that $C \succ_i \pi_i^*$ for all $i \in C$.

A particular kind of blocking coalitions are singletons coalitions $\{i\}$, where i prefers being alone to being in the coalition to which she is matched by the mechanism. Given any profile $\succ \in D$, a partition π^* is **individually rational** if, for all $i \in N$, $\pi_i^* = \{i\}$ or $\pi_i^* \succ_i \{i\}$. Mechanism M is **individually rational** if $M(\succ)$ is individually rational for all $\succ \in D$.

Blocking coalitions also exist if the outcome of a mechanism fails to be Pareto optimal. Given any profile $\succ \in D$, a partition π^* is **Pareto optimal** if there exists no other coalition structure that is preferred by every player to π^*. Clearly, a core partition π is Pareto optimal, because any coalition in a partition π' that is preferred by every player to π is a blocking coalition to π. **Mechanism** M **is Pareto optimal** if $M(\succ)$ is Pareto optimal for all $\succ \in D$.

3 Iterated Matching of Soulmates

Given $\succ \in D$, a coalition $C \in 2^N$ is a $\mathbf{1^{st}}$ **order soulmates coalition** if

$$C \succ_i C' \qquad \text{for all } i \in C \text{ and all } C' \in C_i \setminus C. \tag{2}$$

The process of **iterated matching of soulmates** [IMS] involves repeatedly forming (1^{st} order) soulmates coalitions from player sets decreasing in size as coalitions of soulmates are formed.

While no ordering is required in the formation of soulmates coalitions, it is easiest to describe IMS as if it were a dynamic process. In the first round, given a profile of reported preferences, the mechanisms forms soulmates coalitions. The members of these coalitions prefer their assigned coalition to all others. In the second round, the mechanism forms soulmates coalitions among the players who are not assigned to coalitions in the first round, etc.[12] Formally, the process of iterated matching of soulmates is defined as follows.

[11] The terminology *strong Nash equilibrium* was introduced by Aumann (1959).

[12] As Banerjee, Konishi and Sönmez (2001) and Pápai (2004) noticed, IMS is similar in spirit to the famous *top-trading cycle* algorithm (Shapley and Scarf, 1974). In fact, Banerjee,

Round 1: Form 1^{st} order soulmates coalitions (i.e., coalitions satisfying (2)). Denote the collection of these coalitions by $S_1(\succ)$. The set of players who belong to a coalition in $S_1(\succ)$ is denoted by $N_1(\succ)$. These players are called **$\mathbf{1}^{st}$ order soulmates**.

⋮

Round r: Form coalitions of 1^{st} order soulmates among the players who are *not* part of a coalition that forms in any round preceding round r. Call these coalitions **$\mathbf{r}^{th}$ order soulmates coalitions** and denote the collection of these coalitions by $S_r(\succ)$. The set of players who belong to a coalition in $S_r(\succ)$ is denoted by $N_r(\succ)$. These players are called **$\mathbf{r}^{th}$ order soulmates**.

Formally, given any integer r, the rth order soulmates coalitions are the coalitions C that contain no players from $\cup_{j=1}^{r-1} N_j$ and are such that

$$C \succ_i C' \qquad \text{for all } i \in C \text{ and all } C' \in C_i \backslash C \text{ with } C' \cap (\cup_{j=1}^{r-1} N_j) = \emptyset.$$

End: The process ends when no coalitions forms in some round r^*, i.e., $S_{r^*}(\succ) = \emptyset$.

For convenience, we will denote the collection of coalitions $\cup_{j=1}^{r^*-1} S_j(\succ)$ formed by this process as $IMS(\succ)$. We refer to any player who is matched by IMS as a **soulmate**, and to every coalition that forms under IMS as a **soulmates coalition** (or *coalition of soulmates*).

A mechanism M is a **$\mathbf{1}^{st}$ order soulmates mechanism** if for every $\succ \in D$, every 1^{st} order soulmates coalition forms under M (i.e., $S_1(\succ) \subseteq M(\succ)$). Similarly, a mechanism M is a **soulmates mechanism** if for every $\succ \in D$, the coalitions that form under IMS form under M (i.e., $IMS(\succ) \subseteq M(\succ)$).

3.1 An Example

Some famous coalition formation mechanisms are soulmates mechanisms, for example, the *deferred acceptance* [DA] mechanism in two-sided matching. As we show in Section 5, this follows from the well-know fact that DA always select a core partition. In contrast, despite being a 1^{st} order soulmates mechanism, the *immediate acceptance* [IA] mechanism (or *Boston* mechanism Abdulkadiroğlu and Sönmez, 2003) also used in two-sided matching is *not* a soulmates mechanism.

Example 1 (IA is not a soulmates mechanism)**.** Consider the following profile of preferences over partners in a marriage profile (that here we simplify preference notation by focusing on each player's preferences over potential teammates; as

Konishi and Sönmez (2001, Section 6.5) shows that in the context of a housing market (Shapley and Scarf, 1974), if the players are endowed with the appropriate preferences over coalitions of players, then top-trading cycle is equivalent to IMS (in our terminology).

mentioned above, this is equivalent to defining preferences for each player i over coalitions which include i).

$$\begin{array}{llll} w_1: & m_1 & \succ_1^w & m_2 \\ w_2: & m_1 & \succ_2^w & m_2 \\ w_3: & m_2 & \succ_3^w & m_1 \end{array} \qquad \begin{array}{llllll} m_1: & w_1 & \succ_1^m & w_2 & \succ_1^m & w_3 \\ m_2: & w_1 & \succ_2^m & w_2 & \succ_1^m & w_3 \end{array}$$

In the first round of IA, each woman proposes to her favorite man, and each man who has a proposal immediately forms a coalition with the woman they like best among the women who have proposed to them (hence the name "*immediate* acceptance"). Thus, at the end of the first round, coalitions $\{w_1, m_1\}$ and $\{w_3, m_2\}$ have formed. In the subsequent round, there are no more men available to form a coalition with w_2. Thus, the coalition structure selected by IA is $\{\{w_1, m_1\}, \{w_3, m_2\}, \{w_2\}\}$.

In the first round of IMS, only coalition $\{w_1, m_1\}$ forms. Coalition $\{w_2, m_2\}$ forms in the second round followed by coalition $\{w_3\}$ in the last round. Hence, the coalition structure selected by IMS is $\{\{w_1, m_1\}, \{w_2, m_2\}, \{w_3\}\}$ which differs from that selected by IA.

We refer the reader to Supplementary Material, Section A for two additional examples illustrating IMS.

3.2 IMS-complete Profiles

A profile $\succ \in D$ is **IMS-complete** if all players match through IMS. It is not hard to construct IMS-*in*complete profiles (see Example 5). In the literature, two important classes of IMS-complete profiles are : (1) profiles satisfying the common ranking property (Farrell and Scotchmer, 1988)[13]; and (2) profiles satisfying the top-coalition property (Banerjee, Konishi and Sönmez, 2001). A profile satisfies the **common ranking property** if, for any two coalitions, the players in the two coalitions have the same preferences over these two coalitions.[14] A profile satisfies the **top-coalition property** if for every subset $S \subseteq N$, there exists a coalition $C^* \subseteq S$ which is preferred by all its members to any other coalition made of players from S.[15] The common ranking property implies the top-coalition property, which itself implies IMS-completeness. We illustrate the relationship between the three conditions in Figure 1. See Supplementary Materials Section B for a more complete analysis of the relationship between IMS-completeness and other profile conditions in the literature.[16]

[13] Pycia (2012) leverages the fact that a condition similar to common-ranking (and thus IMS-completeness in our terminology) is implied when all preference profiles in a domain are pairwise-aligned and the domain is sufficiently rich. We discuss this further in supplementary materials, subsection B.3.

[14] Formally, there exists an ordering $\rhd$ of 2^N such that for all $i \in N$ and any $C, C' \in C_i$, we have $C \succ_i C'$ if and only if $C \rhd C'$.

[15] See Banerjee, Konishi and Sönmez (2001) for examples of games from the literature that feature profiles satisfying the common ranking and top-coalition properties.

[16] Supplementary materials, Section B discusses several other conditions studied in Bogomolnaia and Jackson (2002) that guarantee the existence of a core coalition structure in

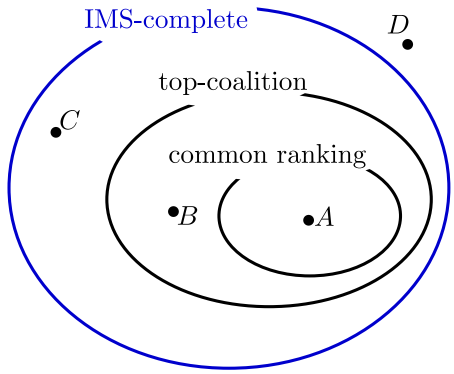

Figure 1: Venn diagram of the profile conditions. A dot indicates that the section of the Venn diagram is non-empty. The inclusion relationship is trivial. For examples of profiles of type A see Banerjee, Konishi and Sönmez (2001, Section 6). For an example of a profile of type B, see Banerjee, Konishi and Sönmez (2001, Game 4). Example 3 in this paper is a profile of type C. Example 5 in this paper is a profile of type D (any other profile for which the core is empty would also be an example).

While profiles satisfying the top-coalition property are IMS-complete the converse is not necessarily true.[17] To gain intuition why, consider a profile in which IMS-completes in two rounds. This implies that there is *a* coalition C_1 and *a* coalition C_2 such that (i) C_1 is a coalition of 1^{st} order soulmates in N, (ii) C_2 is a coalition of 1^{st} order soulmates in $N \backslash C_1$, and (iii) $C_1 \cup C_2 = N$. The top-coalition property is much stronger as it requires that, for *any* coalition C, there be a coalition of 1^{st} order soulmates in $N \backslash C$. This is illustrated more concretely in Example 3, where the profile is IMS-complete but does not satisfy the top-coalition property because there is no coalition of 1^{st} order soulmates in $\{C, G, L\}$.[18]

our environment (including a weakening of top-coalition introduced in Banerjee, Konishi and Sönmez (2001)). Of these conditions only *weak consecutivity* is implied by IMS-completeness. Supplementary materials, Section B also studies three additional conditions for the existence of a core coalition structure introduced by Alcalde and Romero-Medina (2006) and the acyclicity condition introduced by Rodrigues-Neto (2007). IMS-completeness is independent of any of these last conditions.

[17]In Section 6, we present computational results on the size of the overlap between IMS-complete profiles and profiles with the top-coalition property.

[18]Another approach to guarantee that IMS matches all the players is to constrain the set of feasible coalitions. Pápai (2004) shows that if the collection of feasible coalitions satisfies a property she calls *single-lapping*, then IMS matches all players.

4 Incentive Properties of Soulmates Mechanisms

4.1 Incentive Compatibility

As we show in Corollary 1 below, mechanisms that match soulmates have remarkable incentive properties on IMS-complete profiles. Perhaps more significantly, soulmates mechanisms retain these properties *in general* for the players matched by IMS, even in profiles that are *not* IMS-complete.

Any player who reports her preference truthfully and is matched by IMS can find no alternative preference report that makes her better off. In fact, as Proposition 1 below shows, no group of players containing a truth-telling soulmate can collude to simultaneously make themselves better off. Moreover, it follows that the reduced game with player set consisting only of players that are matched as soulmates has a truthful strong Nash equilibrium.

Proposition 1 (No Soulmates Among Deviators)**.** *Suppose that M is a soulmates mechanism. For any reported preference profile $\succ \in D$ and set of players W which contains at least one soulmate and all soulmates in W report their preferences truthfully, there does not exist a joint deviation $\succ'_W$ by the members of W that makes every player in W better-off than reporting $\succ_W$.*

Proof. Recall that $N_1(\succ)$ is the set of 1st order soulmates. According to $\succ$, each member of $N_1(\succ)$ is assigned their most preferred coalition. Thus, if there is a preferred coalition for $i' \in N_1(\succ)$ (and i' could possibly benefit from a deviation), then $\succ_{i'}$ cannot be the true preference of player i'. But then i' cannot be a member of W, since the true preference of every soulmates j in W must be given by $\succ_j$.

The same logic applies to any coalition of soulmates up to the lowest soulmate order, say k^*, which contains a soulmate in W. Now consider a k^{*th} order soulmate i^* who is also a member of W. The coalition of soulmates to which i^* belongs is truly the most preferred coalition to which i^* can belong among all the coalitions that can be formed by players in $N \setminus \cup_{k=1}^{k^*-1} N_k(\succ)$. Thus, i^* cannot benefit from a deviation, which is a contradiction. □

The incentive compatibility implied by the above Proposition does *not* generally extend to players who are not matched by IMS, as we illustrate in the following example.

Example 2 (IMS-serial dictatorship: manipulations in three-sided matching)**.** In **three-sided profiles** (Alkan, 1988) (i) $N = M \cup W \cup D$, (ii) every woman $w \in W$ (resp. man $m \in M$, dog $d \in D$) prefers being in a triple with a man m and a dog d (resp. a woman w and a dog d, a woman w and a man m) to being alone, and (iii) every woman $w \in W$ (resp. man $m \in M$, dog $d \in D$) prefers being alone to being in any coalition different from a triple with a man m and a dog d (resp. a woman w and a dog d, a woman w and a man m).

Consider the following profile of preferences $\succ$ over pairs of partners in a three-sided profile

$$w_1: \quad \{m_1, d_1\} \quad \succ_1^w \quad \{m_1, d_2\} \quad \succ_1^w \quad \{m_2, d_1\} \quad \succ_1^w \quad \{m_2, d_2\}$$

$$
\begin{array}{lccccccc}
w_2: & \{m_1,d_1\} & \succ_1^w & \{m_1,d_2\} & \succ_1^w & \{m_2,d_1\} & \succ_1^w & \{m_2,d_2\} \\
m_1: & \{w_1,d_2\} & \succ_1^m & \{w_2,d_1\} & \succ_1^m & \{w_1,d_1\} & \succ_1^m & \{w_2,d_2\} \\
m_2: & \{w_1,d_2\} & \succ_1^m & \{w_1,d_1\} & \succ_1^m & \{w_2,d_1\} & \succ_1^m & \{w_2,d_2\} \\
d_1: & \{w_2,m_2\} & \succ_1^d & \{w_2,m_1\} & \succ_1^d & \{w_1,m_1\} & \succ_1^d & \{w_1,m_2\} \\
d_2: & \{w_2,m_2\} & \succ_1^d & \{w_1,m_1\} & \succ_1^d & \{w_2,m_1\} & \succ_1^d & \{w_1,m_2\}
\end{array}
$$

Notice that there are no soulmates in this profile: both women want to be matched with $\{m_1, d_1\}$, but m_1 and d_1 do not agree on the most preferred triple. Because there are no soulmates, $IMS(\succ) = \emptyset$.

Consider for instance IMS-serial dictatorship [IMS^{SD}] which consists in first applying IMS and then using the serial dictatorship mechanism to determine the assignment of the players who do not match under IMS. Suppose that the series of dictator is $w_1, w_2, m_1, m_2, d_1, d_2$. Because $IMS(\succ) = \emptyset$, we have $IMS^{SD}(\succ) = SD(\succ)$, the outcome of the serial dictatorship mechanism given profile $\succ$. Thus, $IMS^{SD}(\succ) = \big\{\{w_1, m_1, d_1\}, \{w_2, m_2, d_2\}\big\}$. Man m_1 and dog d_1 can jointly deviate when others report their true preferences by reporting (resp.) $\{w_2, d_1\}$ and $\{w_2, m_1\}$ as their most preferred pairs of partners. This forces IMS^{SD} to form $\{w_2, m_1, d_1\}$ as a coalition of soulmates.

In the above profile, no *single* player can deviate given that other players report their true preference. However, if we replace m_1's preference by

$$
m_1: \quad \{w_2,d_1\} \;\; \tilde{\succ}_1^m \;\; \{w_1,d_2\} \;\; \tilde{\succ}_1^m \;\; \{w_1,d_1\} \;\; \tilde{\succ}_1^m \;\; \{w_2,d_2\}
$$

then, given that other players report their true preference, d_1 can benefit from deviating by reporting $\{w_2, m_1\}$ as her most preferred pair of partners.

There also exist IMS-*in*complete profiles in which no coalition can deviate in IMS^{SD}. This is the case, for example, if the preferences of w_2, m_1 and d_1 are replaced by the following preferences

$$
\begin{array}{lccccccc}
w_2: & \{m_1,d_1\} & \hat{\succ}_1^w & \{m_2,d_2\} & \hat{\succ}_1^w & \{m_2,d_1\} & \hat{\succ}_1^w & \{m_1,d_2\} \\
m_1: & \{w_1,d_2\} & \hat{\succ}_1^m & \{w_1,d_1\} & \hat{\succ}_1^m & \{w_2,d_1\} & \hat{\succ}_1^m & \{w_2,d_2\} \\
d_1: & \{w_2,m_2\} & \hat{\succ}_1^d & \{w_1,m_1\} & \hat{\succ}_1^d & \{w_2,m_1\} & \hat{\succ}_1^d & \{w_1,m_2\}
\end{array}
$$

The outcome of IMS^{SD} is unchanged under $\hat{\succ}$. Because m_1 and d_1 are matched with their second most preferred coalition, the only possible deviations involve forming coalitions $\{w_1, m_1, d_2\}$ or $\{w_2, m_2, d_1\}$. But d_2 prefers her coalition under IMS^{SD} to $\{w_1, m_1, d_2\}$ and w_2 prefers her coalition under IMS^{SD} to $\{w_2, m_2, d_1\}$.

For IMS-complete profiles, however, the following is a corollary of Proposition 1.

Corollary 1 (Truthful Strong Nash Equilibrium)**.** *Suppose that M is a soulmates mechanism. For any IMS-complete profile $\succ$, the preference revelation game $(M, \succ)$ has a truthful strong Nash equilibrium.*

In particular, a soulmates mechanism M always has a truthful strong Nash equilibrium on a domain R containing only IMS-complete profiles. If in addition R is Cartesian and $R = D$, then M is in fact group-strategy proof.

The following Corollary is another interesting and immediate consequence of Proposition 1. A group of players W have no incentive to collude in deviating from truth-telling even if they are uncertain of the preferences of $N \backslash W$, but agree that, in any case, they will all be matched by IMS.

Corollary 2. *Given a mechanism M, suppose that for some player i and some subset of subprofiles $D'_{-i} \subseteq D_{-i}$ it holds that $M_i(\succ_i, \succ'_{-i})$ is a soulmates coalition for all $\succ'_{-i} \in D'_{-i}$ where $\succ_i$ is the true preferences of player i. Then there is no jointly profitable deviation from the profile $(\succ_i, \succ'_{-i})$ for any set of players W containing i and any $\succ_{-i} \in D'_{-i}$.*

For instance, suppose player i is participating in a job-market matching mechanism in which at most one employee will be matched with each employer and there are at least as many employers as job candidates. Suppose it is common knowlege that all employers have a common ranking of employees. (Or, less precisely, suppose that i believes that all employers have a common ranking of job candidates). In terms of the notation of the Corollary, suppose that $\mathcal{D}'_{-i}$ has the property that employers have a common ranking of job candidates. It is easy to see that any preference profile in which employers report a common ranking is IMS-complete.[19] Thus, i can believe that she will be matched by IMS with probability 1 and, for her, truth-telling is a best response. Moreover, from the Proposition, she cannot be convinced to participate in any joint deviation.[20]

4.2 Impossibilities

Our next result shows that the remarkable incentive properties identified in Proposition 1 cannot generally be strengthened much further. In Example 2, we showed that for *some* soulmates mechanisms M, when $\succ$ is not IMS-complete the game $(M, \succ)$ fails to have a truthful Nash equilibrium. Proposition 2 below shows that if R is sufficiently rich, *any* soulmates mechanism M will fail to induce a truthful Nash equilibrium in game $(M, \succ)$ for some IMS-*in*complete $\succ \in R$.

A similar incompatibility can be found in Takamiya (2012, Proposition 3), which shows that, without further restrictions, no two-sided matching mechanism that (in our terminology) is also a 1^{st} order soulmates mechanism is

[19] In the first round of iterated matching of soulmates all firms will point to the highest raned candidate. That candidate will have her first choice of employer. The process will then repeat itself with the 2nd highest ranked candidate now, in the reduced player set, the highest ranked candidate, and so on untill all candidates are matched.

[20] In a prior version of this paper, we formalized the message of this Corollary in a variation of the model with cardinal utilities and expected utility maximization and obtained a formalization of the result that, if player i believes that, with the reported preferences of other players, she will be matched as a soulmate with probability 1 then she will not be willing to join any coalitional deviation.

strategy-proof. As we demonstrate below, this impossibility extends to many domains outside of two-sided matching.

To illustrate this impossibility, consider the following profile of preferences over partners in a roommates profile.

$$\begin{array}{lccccc} 1: & 2 & \succ_1 & 3 & \succ_1 & 1 \\ 2: & 3 & \succ_2 & 1 & \succ_2 & 2 \\ 3: & 1 & \succ_3 & 2 & \succ_3 & 3 \end{array} \tag{3}$$

Suppose that M is a 1^{st} order soulmates mechanism. Mechanism M can form at most one of the three pairs $\{1,2\}$, $\{2,3\}$, $\{3,1\}$. But then, any player who is not in one of these pairs can manipulate by reporting that she is the soulmate of one of the others. Hence, no 1^{st} order soulmates mechanism M has a truthful Nash equilibrium on a domain that includes the above profile (and that allows the aforementioned deviations).

The reason the above profile prevents 1^{st} order soulmates mechanisms to have a truthful Nash equilibrium is that it contains a cycle : 1 likes 2 best, 2 likes 3 best, and 3 likes 1 best. In roommates problems, this impossibility can *only* occur in profiles featuring such cycles. As Rodrigues-Neto (2007) argued, roommates profiles that do not feature cycles of any length are IMS-complete and therefore have a truthful strong Nash equilibrium by Corollary 1.

It is possible to extend the cycle condition from Rodrigues-Neto (2007) to general coalition formation environments. In a general coalition formation environment, soulmates mechanisms may fail to have a truthful Nash equilibrium for reasons other than cycles, but the presence of cycle of odd size is sufficient to induce the impossibility.

The generalized cycle conditions are defined formally in Supplementary Materials, Section C. Intuitively, **individually cyclic** domains have cycles in which coalitions $\{1,2\}$, $\{2,3\}$ and $\{3,1\}$ are replaced in (3) by coalitions of the form $\{1,2\} \cup O_{12}$, $\{2,3\} \cup O_{23}$ and $\{3,1\} \cup O_{31}$, where the O_{jh} are set of players that are allowed to rank coalition $\{j,h,O_{jh}\}$ as their best coalition. In Supplementary materials, Section C, we also define **cyclic** domains in which 1, 2 and 3 are replaced by groups of players N_1, N_2, and N_3 that can jointly deviate. **Individually odd-cyclic** and **odd-cyclic domains** have cycles of the corresponding type that involve an odd number of coalitions.

Proposition 2 (Impossibilities)**.** *(i) If R is an odd-cyclic domain, no 1^{st} order soulmates mechanism M has a truthful* strong *Nash equilibrium on R. (ii) If R is an individually odd-cyclic domain, no 1^{st} order soulmates mechanism M has a truthful Nash equilibrium on R.*

The proof of Proposition 2 generalizes the logic of the above argument for roommates problem (3) and can be found in Supplementary Materials, Section C.

5 Outcome Properties of Soulmates Mechanisms

We now turn to the properties of the outcomes of IMS and soulmates mechanisms with respect to *reported* preferences, focusing on the core and the properties of blocking coalitions.

Proposition 3 (Soulmate Coalitions are Core Coalitions)**.** *Given any $\succ \in D$, let π^* be a partition in the core and let C be a coalition formed by $IMS(\succ)$. Then $C \in \pi^*$.*

Proof. Again, we use the notation in the definition of IMS. Let π^* be a partition in the core and suppose that $C^1 \in S_1(\succ)$, the set of 1^{st} order soulmates coalitions under $IMS(\succ)$. Suppose that $C^1 \notin \pi^*$. Then, from the definition of 1^{st} order soulmates, since each member of C^1 would strictly prefer to be in C^1 rather than in any other coalition, C^1 can improve upon π^*; that is, $C^1 \succ_i \pi^*_i$ for all $i \in C^1$. But since π^* is in the core, this is a contradiction; therefore $C^1 \in \pi^*$. Now consider the player set $N \backslash C^1$ and suppose that $C^2 \in S_2(\succ)$. The coalition C^2 can improve upon any partition of $N \backslash C^1$ that does not contain C^2. This process can be continued until no more players can be matched by IMS. □

Proposition 3 does not rule out the possibility that the core is empty. If the core is empty, then there are no core partitions π^* and Proposition 3 is trivially true. When the core is non-empty, however, the following is a corollary of Proposition 3. Let $I(\succ)$ be the **invariant portion of the core**, i.e., the collection of coalitions that belong to every core partitions given $\succ$. Then, because every player belongs to *at most* one coalition in $IMS(\succ)$, we have the next result.

Corollary 3 (Soulmate Coalitions are Invariant Core Coalitions)**.** *Given any $\succ \in D$, if a core partition exists, then the collection of coalitions $IMS(\succ) \subseteq I(\succ)$.*

In this sense, $IMS(\succ)$ captures a part of the invariant portion of the core. Observe that, by Corollary 3, if there are multiple core partitions but $I(\succ) = \emptyset$ (i.e., no player matches with the same coalition in every core partition), then $IMS(\succ) = \emptyset$.[21] Also observe that, if mechanism M always selects a core outcome when one exist, $I(\succ) \subseteq M(\succ)$ for all $\succ \in D$. Thus, by Corollary 3, $IMS(\succ) \subseteq M(\succ)$ for any such mechanism M (examples include the famous *deferred acceptance* mechanism in two-sided matching).

Theorem 2 in Banerjee, Konishi and Sönmez (2001) shows that the top-coalition property is sufficient to guarantee the existence of a unique core coalition structure. As noted by Banerjee, Konishi and Sönmez (2001), their proof of Theorem 2 generalizes to the case of IMS-complete profiles (in our terminology), and we have the following proposition.[22]

[21] See, for example, the *Latin Square profile* (Van der Linden, 2016) in Klaus and Klijn (2006, Example 3.7).

[22] Theorem 2 as stated in Banerjee, Konishi and Sönmez (2001) does not imply Proposition 4. However, the proof of Theorem 2 as stated in Banerjee, Konishi and Sönmez (2001) also proves Proposition 4 as the authors underline.

Proposition 4 (Unique Core under IMS-complete Profiles)**.** *For every IMS-complete profile $\succ \in D$, the coalition structure $IMS(\succ)$ is the unique core coalition structure.*

Proposition 4 can also be viewed as a consequence of Corollary 3: If IMS successfully matches all players, the entire match must be a part of any core coalition structure, implying that it is the unique core coalition structure. Obviously, this implies that any soulmates mechanism M select the unique core coalition structure for every IMS-complete profile. This also implies that, for any profile $\succ \in D$, mechanism M selects the unique core coalition in the reduced game in which the player set is shrunk to $IMS(\succ)$ itself.

Example 3 and Proposition 4 proved that the top-coalition condition is sufficient but not necessary for the core to be non-empty. The same is true of IMS-completeness. Although, by Proposition 4, IMS-completeness guarantees that the core is non-empty, IMS-completeness is not a necessary conditions for the core to be non-empty, as the next example shows. We demonstrate this in Supplementary Material, Section B.

As for incentives, soulmates mechanisms retain part of their stability properties on IMS-incomplete profiles. Although coalitions can block the outcome of a soulmate mechanism in IMS-*in*complete profiles, these coalitions can only consist of players that do not match in IMS.

Proposition 5 (No Soulmates Among Blockers)**.** *Suppose that M is a soulmates mechanism. For any profile $\succ \in D$, any blocking coalition to $M(\succ)$ contains only players who are* not *soulmates.*

Proof. Again, we use the notation in the definition of IMS. Clearly, no coalition W_1 that blocks $M(\succ)$ contains any players from $N_1(\succ)$, the set of 1^{st} order soulmates.

Now suppose that a coalition W_2 that blocks $M(\succ)$ contains a player from $N_2(\succ)$, but contains no player from $N_1(\succ)$. Then, for any player $i^* \in W_2 \cap N_2(\succ)$, there must exist a coalition $C^* \in \boldsymbol{C}_{i^*}$ such that (i) $C^* \subseteq W_2$, and (ii) $C^* \succ_{i^*} M_{i^*}(\succ)$. However, by definition of IMS, coalition $M_{i^*}(\succ)$ is i^*'s most preferred coalition among the coalitions made of players in $N \backslash N_1(\succ)$. Hence, by (ii), C^* must contain at least one player from $N_1(\succ)$, which contradicts (i). The same logic extends by induction to soulmates of any order. □

The following are corollaries of Propositions 5.

Corollary 4 (Individual Rationality And Pareto Optimal)**.** *Given any IMS-complete profile $\succ$, the coalition structure $IMS(\succ)$ is individually rational and Pareto optimal.*

Corollary 5 (Partial Individual Rationality and Pareto Efficiency)**.** *Suppose that M is a soulmates mechanism. Given any profile $\succ \in D$,*

(i) any player i who is matched with a coalition that she likes less than $\{i\}$ is not a soulmate, and

(ii) for any subset of players $S \subseteq N$ such that there exists a partition π^S of S with $\pi^S_i \succ_i M_i(\succ)$ for all $i \in S$, subset S contains no soulmates.

6 How Common are Soulmates?

How common are IMS-complete profiles? How likely are players to be matched into a coalition by IMS? In this section, we study these questions using empirical analysis of real-world data. We demonstrate that IMS matches many players in three real-world datasets concerning environments where reciprocal or commonly-ranked preferences are at least intuitively likely.

In Supplementary Material, Section C.1 we provide some computational experiments. We focus primarily on the roommates environment (with an even number of players) where every player prefers being in any coalition of two better to being alone. While IMS-complete profiles are relatively rare (in large player sets) when the set of profiles is unconstrained, there is often some structure to preference profiles encountered in real-world problems.

Two particularly natural properties of preference profiles are reciprocity, where individual preferences for others are mutually correlated, and common ranking, in which individual preferences over coalitions are correlated. Our computational experiments indicate that when preferences are highly reciprocal, or close to commonly ranked, IMS matches many players on average and profiles are often IMS-complete.

6.1 Soulmates in the Field

Here we present our empirical results on soulmates in applied problems using real-world datasets in three different environments.

We first consider a roommates problem using social-network data from 1350 students at a university. We next consider a similar problem involving matching coalitions of no more than two in a work setting using data from 44 consultants. Finally, we consider a two-sided matching problem using data from 551 people attending "speed dating" events.

Each data set includes information we use as a surrogate for preferences.[23] In each case, ties occur in the preferences we derive from the data. In our associated working paper Leo et al. (2016) we demonstrate that many of the results for IMS with strict preferences hold when there are indifferences, but that the number of soulmates matched depends on how ties are broken. Because of this, for each set of data, we have run IMS 10,000 times with random tie-breaking each time and recorded the proportion of players matched in each case. More details about the data sets and the precise assumptions used in deriving preferences are given below.

In each environment IMS was able to match about a third of players on average and sometimes substantially more depending on the tie-breaking- up to three-quarters in one instance of the work teams data. This is far more than would be predicted by our results on unstructured preferences in section C.1.

[23] The data in this section were not used to produce an actual matching. Thus, it is less likely that the derived preferences are already strategic.

To illustrate better how IMS is operating in these environments, the table below details the number of individuals left after each round of IMS (for a single random tie-breaking of preferences). This demonstrates that the iterated nature of IMS is far from trivial in practice. In each case, IMS is able to match at least 6^{th} order soulmates and, for example, in the roommates data there are 34 5^{th} order soulmates for this particular tie-breaking of preferences.

	Roommates	Work Teams	Dating (Overall)	Dating (Shared Interest)
Start	1350	44	551	551
1	1116	36	469	471
2	1000	34	431	435
3	936	32	407	413
4	896	30	395	397
5	862	28	387	385
6	840	**26**	385	375
7	826		**383**	371
8	824			**369**
9	**822**			

Table 1: Group Size Remaining After Each Round of IMS. Bold numbers indicate the final size at the end of IMS.

6.1.1 University Roommates

Our roommates dataset comes from a network of 1,350 users[24] of a "Facebook-Like" social network at the University of California Irvine. The data is provided by and described in Panzarasa, Opsahl and Carley (2009). The data includes, for each user, the number of characters sent in private messages to each other user. We use this information as a surrogate for the preference data of each user, assuming that if a user sends more characters to i than to j than the user would prefer to be matched with i over j. We assume that a user would prefer to be matched with any random partner than to remain alone.

Over 1,000 trials, an average of 39.0% of users are matched by IMS. The maximum was 39.4%. Since the data measures characters sent- a relatively fine-grained measure, most of the ties occur where the users have sent zero characters to each other. This has the effect of randomizing the "bottom" of each users preference list and does not substantially affect on IMS. The success of IMS in this case is likely due to the reciprocity in the derived preferences. If i sends many messages to j, then it is likely that j sends many to i.

[24]The dataset contains 1,899 users but only 1,3500 have the message data we use to produce preferences.

6.1.2 Work Teams

The work coalitions data set comes from a study of 44 consultants within a single company. The data is provided by and described in Cross et al. (2004). The consultants responded on a 1-5 scale for each of the other consultants to the question "In general, this person has expertise in areas that are important in the kind of work I do." Here, we assume that if a consultant gave a higher score to i than to j, the consultant would rather be on a coalition with i. (Again, ties in preferences are broken randomly.) We again assume that a consultant would prefer to be matched with any random partner than to remain alone.

Over 1,000 trials, an average of 31.3% of consultants are matched by IMS. The maximum was 77.3%. Since the preference measure is less fine-grained in this case, tie-breaking randomization has a stronger effect. Here, it is likely that elements of reciprocity and common-ranking are present in the data. Expertise is a relatively objective measure, while the fact that the question asks about "work I do" makes two consultants with the same focus likely to give each other higher scores.

6.1.3 Speed Dating

The speed dating dataset come from a study of 551 students at Columbia University invited to participate in a speed dating experiment. The data is provided by and described in Fisman et al. (2006). Each participant had a four-minute conversation with roughly 10-20 partners and was then asked to rate each partner on a 1-10 scale in various aspects. Here, we focus on two ratings: "Overall, how much do you like this person?" and a shared interest rating. For both, we assume that if a participant gave a higher ranking to i than to j this participant would rather be matched with i than with j.

The proportion matched depends on the question. On average 26.2% can be matched by IMS based on the overall ranking but 31.0% can be matched using the shared interest rating. The maximum for the overall rating was 36.6% and 43.2% for shared interest. The improvement of using shared interest is likely due to the additional reciprocal structure in shared interest data.

In Supplementary Material, Section C.1, we ccompare the proportion of IMS-complete profiles to the proportions of other types of profiles that imply IMS-completeness, some of which are discussed in the literature: top-coalition, common-ranking, and reciprocal profiles.

7 Conclusion

In this paper, we have provided the first detailed study of iterated matching of soulmates (IMS). Mechanisms which implement IMS provide very desirable properties among players who are matched by IMS. These players have no incentive to deviate from truth-telling, even jointly, and their matched coalitions are part of any in any core match. Indeed, for these players (and players who *believe* they will be matched by IMS), truth-telling is a dominant strategy.

While IMS presents opportunities for manipulation by those not matched into a coalition of soulmates, under some natural types of preferences many players – and often the entire player set – can be matched by IMS. This suggests that while there are generally many trade-offs to consider in designing coalition formation mechanisms, when preferences are likely to be approximately commonly ranked, reciprocal, or otherwise IMS-complete, soulmates mechanisms provide desirable outcomes with little sacrifice.

This highlights an interesting avenue for further study: What mechanisms appropriate for the general matching environment – or even specialized to particular environments – implement IMS and handle IMS-incomplete profiles while making minimal sacrifices in terms of desirable properties? More generally, the results here hint at some deep tensions between optimality and incentives in the general matching environment. These tensions deserve further, direct investigation.

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Supplementary Materials (For Online Publication)

A Some Examples

Example 3 (Formation of parliamentary groups)**.** A Left (L), a Center (C), a Right (R) and a Green (G) party have to form parliamentary groups. Their preferences form a **roommates profile** : (i) every party prefers a coalition of two to being alone, and (ii) every party prefers being alone to being in a coalition of more than two players. The parties have the following preferences over partners

$$\begin{array}{lccccc} R: & C & \succ_R & G & \succ_R & L \\ C: & R & \succ_C & G & \succ_C & L \\ G: & L & \succ_G & C & \succ_G & R \\ L: & C & \succ_L & G & \succ_L & R \end{array}$$

If we apply IMS to this profile, coalition $\{R, C\}$ forms in the first round, and coalition $\{G, L\}$ forms in the second round.

Example 4 (Marriage, aligned women and cyclic men)**.** In a **marriage profile** (i) $N = M \cup W$, (ii) every woman $w \in W$ (resp. man $m \in M$) prefers being in a pair with a man m (resp. woman w) to being alone, and (iii) every woman $w \in W$ (resp. man $m \in M$) prefers being alone to being in any coalition different from a pair with a man m (resp. woman w). Consider the following profile of preferences over partners

$$\begin{array}{lccccc} w_1: & m_1 & \succ_1^w & m_2 & \succ_1^w & m_3 \\ w_2: & m_1 & \succ_2^w & m_2 & \succ_2^w & m_3 \\ w_3: & m_1 & \succ_3^w & m_2 & \succ_3^w & m_3 \end{array} \qquad \begin{array}{lccccc} m_1: & w_1 & \succ_1^m & w_2 & \succ_1^m & w_3 \\ m_2: & w_2 & \succ_2^m & w_3 & \succ_2^m & w_1 \\ m_3: & w_3 & \succ_3^m & w_1 & \succ_3^m & w_2 \end{array}$$

In the first round of IMS, $\{m_1, w_1\}$ forms. In the second round, given that m_1 has already been matched, $\{m_2, w_2\}$ is a coalition of soulmates and forms. In the third round, given that m_1 and m_2 have already been matched $\{m_3, w_3\}$ is a coalition of soulmates and forms. It is easy to see how this example extends to larger sets of players.

Example 5. Consider the following profile of preferences over partners in a roommates profile.

$$\begin{array}{lccccc} 1: & 2 & \succ_1 & 3 & \succ_1 & 4 \\ 2: & 4 & \succ_2 & 1 & \succ_2 & 3 \\ 3: & 2 & \succ_3 & 1 & \succ_3 & 4 \\ 4: & 1 & \succ_4 & 2 & \succ_4 & 3 \end{array}$$

Recall that, by definition of a roommates profile, every player prefers a coalition of two to being alone and prefers being alone to being in a coalition of more than two players and . Therefore, because there is an even number of players, no player is alone in a core coalition structure.

If 1 matches with 2 then 3 match with 4 in which case 2 and 4 can deviate. If 1 matches with 4 then 2 matches with 3 in which case 1 and 4 can deviate. Thus 1 must match with 3 in any core coalition structure. This leaves coalition structure $\{\{1,3\},\{2,4\}\}$ which is a core coalition structure, and hence the unique core coalition structure. Clearly, IMS does not match all the players under this profile as there are no soulmates in N.

Observe that, when players manipulate their preferences and report, for example, the profile of aligned preferences $\succ'$ with $1 \succ'_i 2 \succ'_i 3 \succ'_i 4$ for all $i \in \{1,2,3,4\}$, any soulmates mechanism would select the core partition $\{\{1\},\{2\},\{3\},\{4\}\}$ (recall that the core is defined with respect to the *reported preferences*). However, $\{\{1\},\{2\},\{3\},\{4\}\}$ is not a core partition with respect to the true profile $\succ$ because any pair of players is a blocking coalition. This illustrates the importance of truthfulness incentives for the properties of outcomes to be relevant with respect to the true preferences (Section A).

B Discussion of the relationship between IMS-completeness and other profile conditions in the literature

B.1 Weak consecutiveness, weak top-coalition, and ordinal balancedness

In this appendix, we analyze the relationships between IMS-completeness and three additional profile conditions notably studied in Bogomolnaia and Jackson (2002) and Banerjee, Konishi and Sönmez (2001). These relationships are represented in Figure 2 and Figure 3. Let us first define these additional profile conditions.

Greenberg and Weber (1986) introduced the concept of a consecutive profile (or game) for transferable utility games.[25] Bogomolnaia and Jackson (2002) adapt it to the non-transferable case. The following follows Bogomolnaia and Jackson (2002). A coalition $C \in 2^N$ is **consecutive with respect to an ordering** $\trianglerighteq$ of the players in N if for all $j,k,h \in N$, $[j \in C,\ h \in C$ and $j \triangleright k \triangleright h]$ implies $k \in C$. A profile $\succ$ is **weakly consecutive** if there exists an ordering $\trianglerighteq$ of the players such that for every coalition structure π, whenever a coalition S blocks π, there exists a consecutive coalition S' that also blocks π.

Shapley and Scarf (1974) introduced the concept of an ordinally balanced profile, again in the context of games of transferable utility. Bogomolnaia and Jackson (2002) and Banerjee, Konishi and Sönmez (2001) propose equivalent adaptation to the non-transferable case. Here we follow Bogomolnaia and Jackson (2002) in terms of exposition. A collection of coalitions $C \subseteq 2^N$ is **balanced** if there exists a vector d of positive weights d_C such that for each player $i \in N$,

[25] Games with consecutive coalitions have non-empty cores independently of the payoff functions. Kaneko and Wooders (1982) provides a characterization of conditions on admissible coalition structures for transferable and nontransferable games to have this property.

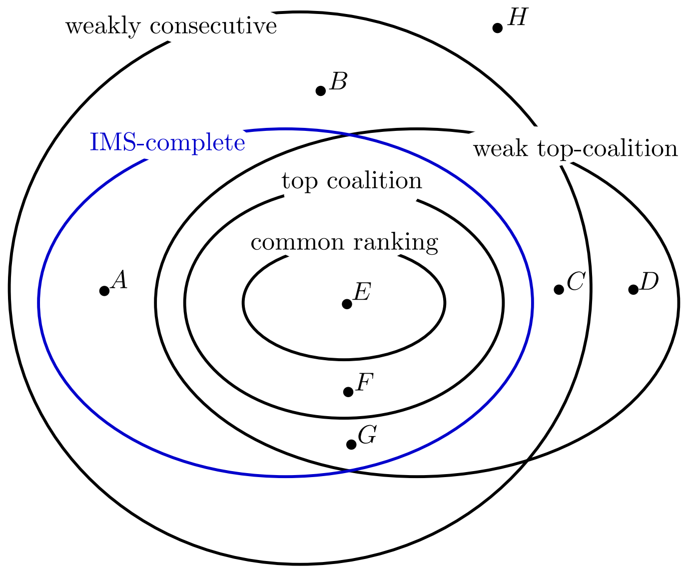

Figure 2: Venn diagram of the relationship between IMS-complete, weak top-coalition and other profile conditions in the literature (when preferences are strict). A dot indicates that the section of the Venn diagram is non-empty.

we have $\sum_{\{C \in \mathcal{C} | i \in C\}} d_C = 1$. As Bogomolnaia and Jackson (2002) put it, a profile is ordinally balanced if for each balanced collection of coalitions, there exists a coalition structure such that each player weakly prefers her coalition in the coalition structure to her *worst* coalition in the balanced collection. Formally, a profile $\succ$ is **ordinally balanced** if for each balanced collection of coalitions $\mathcal{C} \subseteq 2^N$, there exists a coalition structure π such that for each $i \in N$, there exists $C \in \mathcal{C}$ with $i \in C$ such that $\pi_i = C$ or $\pi_i \succ_i C$.

Finally, Banerjee, Konishi and Sönmez (2001) introduce a weaker version of the top-coalition property. Given a set of players $S \in 2^N$, a coalition $C \subseteq S$ is a **weak top-coalition** of S if and only if C has an ordered coalition structure $\{C^1, \ldots, C^l\}$ such that (i) for any $i \in C^1$ and any $T \subseteq S$ with $i \in T$, we have $C \succ_i T$ and (ii) for any $k > 1$, any $i \in C^k$ and any $T \subseteq S$ with $i \in T$, we have $[T \succ_i C] \Rightarrow \left[T \cap \left(\bigcup_{m<k} S^m\right) \neq \emptyset\right]$. A profile $\succ$ satisfies the **weak top-coalition property** if and only if for any non-empty set of player $S \subseteq N$, there exists a weak top-coalition C.

Any of these three additional properties is sufficient for the existence of a core coalition structure (Banerjee, Konishi and Sönmez (2001), Bogomolnaia and Jackson (2002)).[26] We now prove every relationship in Figure 2.

[leftmargin=*]

[26] See also Dimitrov (2006) for a the relation between the top coalition property and a stronger stability condition dubbed the *semistrict core*.

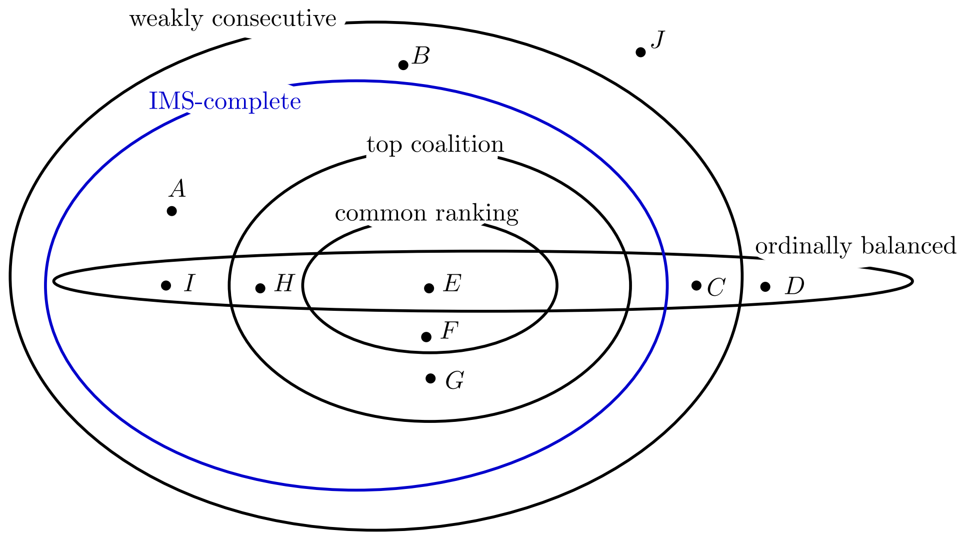

Figure 3: Venn diagram of the relationship between IMS-completeness, ordinal balancedness and other profile conditions in the literature (when preferences are strict). A dot indicates that the section of the Venn diagram is non-empty.

IMS-complete ⇒ weakly consecutive. See the first part of the proof of Proposition 1 in (Bogomolnaia and Jackson, 2002, p.211) which proves that the top-coalition property implies the weakly consecutive property. The proof of this result is easily adapted to show that IMS-completeness implies the weakly consecutive property.

A. *IMS-complete, not Weak Top Coalition*

Consider the following profile of preferences over partners in a roommates profile

$$\begin{array}{lccccc} 1: & 2 & \succ_1 & 3 & \succ_1 & 4 \\ 2: & 1 & \succ_2 & 3 & \succ_2 & 4 \\ 3: & 1 & \succ_3 & 4 & \succ_3 & 2 \\ 4: & 1 & \succ_4 & 2 & \succ_4 & 3 \end{array}$$

The profile is IMS-complete with $\{\{1,2\}\{3,4\}\}$ as the outcome of IMS, but it does not satisfy any of the properties (i) common ranking, (ii) top coalition, or (iii) weak-top coalition, where the fact that the profiles satisfies neither the top coalition property nor the weak-top coalition property follows from $\{2,3,4\}$ not having a top-coalition or a weak top-coalition.

B. *Weakly Consecutive, not IMS-complete, not Weak Top Coalition*

See the second part of the proof of Proposition 1 in (Bogomolnaia and Jackson, 2002, p.212) which proves that the weakly consecutive property does not imply the top coalition property. The example given in that proof is weakly consecutive, but fails (i) to be IMS-complete, (ii) to be ordinally balanced, and (iii) to satisfy the weak top-coalition property.

C. *Weakly Consecutive, Weak Top Coalition, not IMS-complete*

Consider the following profile of preferences over coalitions, where ... indicates that the rest of the preferences are arbitrary

$$\begin{array}{llllllll} 1: & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \dots & & \\ 2: & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \dots & & \\ 3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \dots \\ 4: & 1 & \succ_4 & 2 & \succ_4 & 3 & & \end{array}$$

Let the ordering of the players be $1 \rhd 2 \rhd 3$. Every coalition structure except $\{\{1,3\},\{2\}\}$ is blocked by $\{1,2,3\}$, which is consecutive according to $\rhd$. Partition $\{\{1,3\},\{2\}\}$ is blocked by $\{1\}$ which is consecutive too. Hence every coalition structure that is blocked is also blocked by a consecutive coalition and the profile is weakly consecutive (the core coalition structure is N).

Also, $\{1,2,3\}$ is a weak top-coalition for $\{1,2,3\}$, and every other set of players admits one of its singletons as a weak top-coalition (e.g. $\{1,3\}$ has $\{1\}$ as a weak top-coalition). Hence the profile satisfies the weak top-coalition property.

However, N does not have a top-coalition, and the profile is therefore not IMS-complete.

D. *Weak Top Coalition, not Weakly Consecutive*

See the second profile on (Bogomolnaia and Jackson, 2002, p.212). As the authors show, the profile satisfies the weak top-coalition property, but is not weakly consecutive. As N does not have a top-coalition, the profile satisfies neither the top-coalition property nor IMS-completeness.

E. *Common Ranking*

See profiles of type A in Figure 1.

F. *Top Coalition, not Common Ranking*

See profiles of type B in Figure 1.

G. *IMS-complete, Weak Top Coalition, not Top Coalition*

Consider the following profile of preferences over coalitions, where ... indicates that the rest of the preference is arbitrary

$$\begin{array}{llllllllll} 1: & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \dots & & & & \\ 2: & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \dots & & & & \\ 3: & \{3,4\} & \succ_3 & \{2,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \dots \\ 4: & \{3,4\} & \succ_4 & \{4\} & \succ_4 & \dots & & & & \end{array}$$

Coalition $\{1,2,3\}$ is a weak top-coalition for $\{1,2,3\}$. Coalitions $\{1\}$ or $\{2\}$ are weak top-coalitions for any other set of players containing 1 or 2. Finally, $\{3,4\}$ is a weak top-coalition for $\{3,4\}$, and $\{3\}$ and $\{4\}$ are weak top-coalitions for $\{3\}$ and $\{4\}$ respectively. Hence, the profile satisfies the weak top-coalition property.

The profile is IMS-complete, with IMS yielding coalition structure $\{\{1\},\{2\},\{3,4\}\}$. However there is no top coalition for $1,2,3$.

H. *Not Weakly Consecutive, not Weak Top Coalition*
See profiles of type D in Figure 1 (any other profile for which the core is empty would also be an example).

We now turn to the elements of Figure 3.

[leftmargin=*]

IMS-complete ⇒ weakly consecutive. See above.

A. *IMS-complete, not Ordinally Balanced, not Top Coalition*
Consider the following profile of preferences over coalitions, where ... indicates that the rest of the preference is arbitrary

$$\begin{array}{ccccccc} \{1,2\} & \succ_1 & \{1,3\} & \succ_1 & \{1\} & \succ_1 & \dots \\ \{1,2\} & \succ_2 & \{2,4\} & \succ_2 & \{2\} & \succ_2 & \dots \\ \{3,5\} & \succ_3 & \{3,4\} & \succ_3 & \{3\} & \succ_3 & \dots \\ \{1,4\} & \succ_4 & \{4,5\} & \succ_4 & \{4\} & \succ_4 & \dots \\ \{3,5\} & \succ_5 & \{4,5\} & \succ_5 & \{5\} & \succ_5 & \dots \end{array} \tag{4}$$

The profile does not satisfy the top-coalition property because $\{1,3,4\}$ does not have a top-coalition.

The profile does not satisfy ordinal balancedness either, with respect to the balanced collection $BC = \{\{1,2\},\{3,4\},\{3,5\},\{4,5\}\}$. Any partition satisfying the balancedness condition with respect to BC must match $\{1,2\}$ and cannot match $\{3,4,5\}$. But then the player who is left alone among $\{3,4,5\}$ cannot be better-off than in any coalition in BC.

However, the profile is IMS-complete with IMS yielding coalition structure $\{\{1,2\},\{3,5\},\{4\}\}$.

B. *Weakly Consecutive, not IMS-complete, not Ordinally Balanced*
See the first profile on (Bogomolnaia and Jackson, 2002, p.212). As the authors show, the profile is weakly consecutive, but violates ordinal balance. The profile fails to be IMS-complete as N does not have a top-coalition.

C. *Weakly Consecutive, Ordinally Balanced, not IMS-complete*
Consider the following profile of preferences over coalitions, where ... indicates that the rest of the preference is arbitrary

$$\begin{array}{lccccccc}
1: & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \ldots & & \\
2: & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \ldots & & \\
3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \ldots \\
4: & 1 & \succ_4 & 2 & \succ_4 & 3 & &
\end{array}$$

For every balanced collection that contains $\{1,2,3\}$, coalition structure $\{1,2,3\}$ is such that every player likes a coalition in the balanced collection (namely $\{1,2,3\}$) at least as much as the coalition structure. The remaining balanced collection are (i) $\{\{1\},\{2\},\{3\}\}$, (ii) $\{\{1,2\},\{3\}\}$, (iii) $\{\{1\},\{2,3\}\}$, (iv) $\{\{1,3\},\{2\}\}$, and (v) $\{\{1,2\},\{2,3\},\{1,3\}\}$. For (i) to (iv), the balanced collection is itself a coalition structure. For (v), coalition structure $\{1,2,3\}$ is again such that every player likes $\{1,2,3\}$ better than some coalition in $\{\{1,2\},\{2,3\},\{1,3\}\}$. Hence, the profile is balanced.

However, N does not have a top-coalition and therefore, the profile is IMS-*in*complete.

D. *Ordinally Balanced, not Weakly Consecutive*
Consider the profile at the top of (Bogomolnaia and Jackson, 2002, p.214). As the authors show, the profile is ordinally balanced and is not weakly consecutive.

E. *Common Ranking, Ordinally Balanced*
Profile

$$\begin{array}{cccccc}
1: & \{1,2\} & \succ_1 & \{1\} & \succ_1 & \ldots \\
2: & \{1,2\} & \succ_2 & \{2\} & \succ_2 & \ldots
\end{array}$$

is trivially both a common ranking profile and an ordinally balanced profile.

F. *Common Ranking, not Ordinally Balanced*
See Game 5 in Banerjee, Konishi and Sönmez (2001).

G. *Top Coalition, not Common Ranking, not Ordinally Balanced*
Consider the following profile of preferences over coalitions which is adapted from profile 4.

$$\begin{array}{ccccccc}
\{1,2\} & \succ_1 & \{1,3\} & \succ_1 & \{1\} & \succ_1 & \ldots \\
\{1,2\} & \succ_2 & \{2,4\} & \succ_2 & \{2\} & \succ_2 & \ldots \\
\{3,4\} & \succ_3 & \{3,5\} & \succ_3 & \{3\} & \succ_3 & \ldots \\
\{3,4\} & \succ_4 & \{4,5\} & \succ_4 & \{4\} & \succ_4 & \ldots \\
\{3,5\} & \succ_5 & \{4,5\} & \succ_5 & \{5\} & \succ_5 & \ldots
\end{array}$$

The profile satisfies the top-coalition property. The profile however violates ordinal balancedness for the same reason profile 4 violates ordinal balancedness in A.

H. *Ordinally Balanced, Top Coalition, not Common Ranking*

Consider the following profile of preferences over coalitions, where ... indicates that the rest of the preference is arbitrary

$$\begin{array}{llllll} 1: & \{1,2\} & \succ_1 & \{1\} & \succ_1 & \dots \\ 2: & \{1,2\} & \succ_2 & \{2\} & \succ_2 & \dots \\ 3: & \{1,3\} & \succ_3 & \{3\} & \succ_3 & \dots \end{array}$$

The profile satisfies the top-coalition property with $\{1,2\}$ as the top-coalition for both $\{1,2,3\}$ and $\{1,2\}$, $\{3\}$ as top-coalition for $\{3\}$, $\{2\}$ as top-coalition for $\{2,3\}$ and $\{2\}$, and $\{1\}$ as top-coalition for $\{1,3\}$ and $\{1\}$.

I. *Ordinally Balanced, IMS-complete, not Top Coalition*

Consider the following profile of preferences over coalitions, where ... indicates that the rest of the preference is arbitrary

$$\begin{array}{llllllll} 1: & \{1,2\} & \succ_1 & \{1\} & \succ_1 & \dots & & \\ 2: & \{1,2\} & \succ_2 & \{2,3\} & \succ_2 & \{2\} & \succ_2 & \dots \\ 3: & \{3,4\} & \succ_3 & \{3\} & \succ_3 & \dots & & \\ 4: & \{2,4\} & \succ_4 & \{3,4\} & \succ_4 & \{4\} & \succ_4 & \dots \end{array}$$

The profile is ordinally balanced. For most balanced collections of coalitions, the balancedness condition is satisfied with respect to the core coalition structure $\pi^* = \{\{1,2\},\{3,4\}\}$. If this is not the case for some balanced collections of coalitions, then this collection must contain $\{2,4\}$ and cannot contain any other coalition that includes player 4. But then $\{2,4\}$ must have weight 1 which means the collection does not contain any other coalition that includes player 2 either. Any such balanced collection satisfies the balancedness condition with respect to partition $\pi^{**} = \{\{2,4\},\{1\},\{3\}\}$.

The profile does not satisfy the top-coalition property because $\{2,3,4\}$ does not have a top-coalition.

The profile is IMS-complete with IMS yielding coalition structure $\{\{1,2\},\{3,4\}\}$.

J. *Not Weakly Consecutive, not Ordinally Balanced*

See Example 5 (any other profile for which the core is empty would also be an example).

Let us finally note that ordinal balancedness and the weak top-coalition property are also independent of one another, in the sense that there exists profile satisfying one of the properties but not the other, as proven in (Bogomolnaia and Jackson, 2002, Proposition. 1)

B.2 Alcalde and Romero-Medina (2006)

Alcalde and Romero-Medina (2006) present four conditions which guarantee the existence of a core allocation. Below, we demonstrate there is no containment relation between IMS-complete and any of these four conditions. The definitions below follow Alcalde and Romero-Medina (2006) and are, again, for profiles of strict preferences.

A profile is **union responsive** if for every $i \in N$ and any two coalitions C, C'^N such that $C' \subset C$, and C' is not the most preferred coalition for i in 2^C, we have $C \succ_i C'$.

The following profile is IMS-complete but violates union responsiveness

$$\begin{array}{lccccc} 1: & \{1\} & \succ_1 & \dots & & \\ 2: & \{1,2\} & \succ_2 & \{2,3\} & \succ_2 & \dots \\ 3: & \{1,3\} & \succ_3 & \{2,3\} & \succ_3 & \dots \end{array} \tag{5}$$

For example, $\{2,3\} \subset \{1,2,3\}$, $\{2,3\}$ is not the most preferred coalition for 2 in $2^{\{1,2,3\}}$ and $\{2,3\} \succ_2 \{1,2,3\}$

The following profile is union responsive but not IMS-complete.

$$\begin{array}{lccccccc} 1: & \{1,2\} & \succ_1 & \{1,2,3\} & \succ_1 & \{1,3\} & \succ_1 & \{1\} \\ 2: & \{2,3\} & \succ_2 & \{1,2,3\} & \succ_2 & \{1,2\} & \succ_2 & \{2\} \\ 3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{2,3\} & \succ_3 & \{3\} \end{array} \tag{6}$$

A profile is **intersection responsive** if for every $i \in N$ and any two coalitions C, C'^N, $C \succ_i C'$ implies $C \cap C' \succ_i C'$.

Again, profile (5) is IMS-complete but violate intersection responsiveness. For example, $\{1,2\} \cap \{2,3\} = \{2\}$, $\{1,2\} \succ_2 \{2,3\}$ but $\{2,3\} \succ_2 \{2\}$). The following variant of profile (6) is intersection responsive but not IMS-complete.

$$\begin{array}{lccccccc} 1: & \{1,2\} & \succ_1 & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \{1,3\} \\ 2: & \{2,3\} & \succ_2 & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \{1,2\} \\ 3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \{2,3\} \end{array}$$

A profile is **singular** if for every $i \in N$ there is a unique acceptable coalition $C \in 2^N$ (recall that an acceptable coalition C is a coalition for which $C \succ_i \{i\}$).

Profile (5) is IMS-complete but not singular. The following profile is singular but not IMS-complete.

$$\begin{array}{lccccc} 1: & \{1,2\} & \succ_1 & \{1\} & \succ_1 & \dots \\ 2: & \{2,3\} & \succ_2 & \{2\} & \succ_2 & \dots \\ 3: & \{1,3\} & \succ_3 & \{3\} & \succ_3 & \dots \end{array}$$

Finally, a profile is **essential** if for every $i \in N$, there is an essential coalition $C^i \in 2^N$, that is, a coalition such that

1. [(i)]

2. if $C^i = \{i\}$, then $\{i\} \succ_i C$ for any $C \neq \{i\}$, and
3. if $C^i \neq \{i\}$, then
   (a) $\{i\} \succ_i S$ if and only if S is not a superset of C^i, and
   (b) for any two coalitions C, C'^N, if $C^i \subseteq C \subset C'$, then $C \succ_i C'$.

The following variant of profile (5) is IMS-complete but not essential. For example, neither $\{1,2\}$ nor $\{2,3\}$ are a superset of one another, but both are acceptable for 2.

$$\begin{array}{lccccccc} 1: & \{1\} & \succ_1 & \dots & & & & \\ 2: & \{1,2\} & \succ_2 & \{2,3\} & \succ_2 & \{2\} & \succ_2 & \dots \\ 3: & \{1,3\} & \succ_3 & \{2,3\} & \succ_3 & \{3\} & \succ_3 & \dots \end{array}$$

The following profile is essential but not IMS-complete

$$\begin{array}{lccccccc} 1: & \{1,2\} & \succ_1 & \{1,2,3\} & \succ_1 & \{1\} & \succ_1 & \dots \\ 2: & \{2,3\} & \succ_2 & \{1,2,3\} & \succ_2 & \{2\} & \succ_2 & \dots \\ 3: & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} & \succ_3 & \dots \end{array}$$

B.3 Pycia (2012)

Pycia (2012) studies rules for sharing coalition surplus that guarantee the existence of a stable outcome in a generalized many-to-one matching environment. The relevant condition that Pycia identifies in the induced preferences is that of *pairwise alignment*. A profile is said to be pairwise aligned if for any two $i, j \in N$ and coalitions C and C' both containing i and j, $C \succeq_i C' \Leftrightarrow C \succeq_j C'$

Theorem 1 of Pycia (2012) states that, in this environment, when all potential preference profiles induced by a sharing rule are *pairwise aligned* and the domain of induced preference profiles is sufficiently rich, there is a unique stable outcome when preferences are strict. The proof of uniqueness (see Pycia (2012) *lemma 5*) utilizes the logic of matching soulmates, leveraging the fact that under these conditions, a relaxed version of common-ranking holds which implies that the profile is IMS-complete.

We note that pairwise alignment (in isolation of the additional domain richness requirement) is logically distinct from IMS-completeness. The following profile is IMS-complete (and Top-Coalition) but not pairwise-aligned (due to the preferences of 1 and 2 over {1,2,3} and {1,2}).

$$\begin{array}{lccccccc} 1: & \{1,3\} & \succ_1 & \{1,2,3\} & \succ_1 & \{1,2\} & \succ_1 & \{1\} \\ 2: & \{2,3\} & \succ_2 & \{1,2\} & \succ_2 & \{1,2,3\} & \succ_2 & \{2\} \\ 3: & \{2,3\} & \succ_3 & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} \end{array}$$

The following profile is pairwise-aligned but not IMS-complete (due to the top-coalition cycle):

$$\begin{array}{lllllll} 1: & \{1,3\} & \succ_1 & \{1,2\} & \succ_1 & \{1,2,3\} & \succ_1 & \{1\} \\ 2: & \{1,2\} & \succ_2 & \{2,3\} & \succ_2 & \{1,2,3\} & \succ_2 & \{2\} \\ 3: & \{2,3\} & \succ_3 & \{1,3\} & \succ_3 & \{1,2,3\} & \succ_3 & \{3\} \end{array}$$

This profile demonstrates the importance of the further condition that Pycia's theorem imposes: that the profile is embeddable in a rich-enough domain of pairwise-aligned preferences (see Pycia's assumption *R1*).

B.4 Cyclical roommates profiles Rodrigues-Neto (2007)

For roommates profiles, Rodrigues-Neto (2007) defines an acyclicity condition that strengthens the "no odd rings" condition from Chung (2000). A roommates profile is **acyclic** if there exists no subset of agents $\{i(1), \ldots, i(k)\}$ with $k \geq 3$ and $i(j) \neq i(j+1)$ for all $j \in \{1, \ldots, k-1\}$ such that

$$\begin{aligned} & i(j+1) \succ_{i(j)} i(j-1) \qquad \text{for all } j \in \{2, \ldots, k-1\}, \\ & i(1) \succ_{i(k)} i(j-1), \text{ and} \\ & i(2) \succ_{i(1)} i(k) \end{aligned} \tag{7}$$

As Chung (2000) shows, the "no odd rings" condition is sufficient for the non-emptiness of the core. Because the acyclicity condition from Rodrigues-Neto (2007) strengthens the "no odd rings" condition, it is also sufficient for the core to be non-empty. In fact, as Rodrigues-Neto (2007) argues, if a roommates profile is acyclic, then the profile is also IMS-complete. The next example shows that the converse is not true. Consider the following profile of preferences over partners in a roommates profile

$$\begin{array}{llllll} 1: & 4 & \succ_1 & 2 & \succ_1 & 3 \\ 2: & 4 & \succ_2 & 3 & \succ_2 & 1 \\ 3: & 4 & \succ_3 & 1 & \succ_3 & 2 \\ 4: & 1 & \succ_4 & 2 & \succ_4 & 3 \end{array}$$

Players $1, 2$ and 3 form a cycle but because 1 and 4 are soulmates, the cycle is "broken" after soulmates have been matched together and the profile is IMS-complete with IMS producing partition $\{\{1,4\},\{2,3\}\}$.

C Cyclical domains and proof of Proposition 2

In this appendix, we provide generalizations to the general coalition formation environment of a cycle notion defined by Rodrigues-Neto (2007) for roommates profiles (see (7)). We then use these generalization to prove Proposition 2.

A **coalition k-cycle** is a collection of coalitions

$$\begin{aligned} & \{C_{12}, C_{23} \ldots, C_{k1}\} \text{ such that } C_{ij} = N_i \cup N_j \cup O_{ij}, \\ & \text{for some non-empty } N_1, \ldots, N_k \subset N \text{ with } N_1 \cap N_2 = \cdots = N_{k-1} \cap N_k = \emptyset \end{aligned}$$

and some (possible empty) $O_{12}, O_{23}, \dots, O_{k1} \subset N \backslash (N_1 \cup \dots \cup N_k)$
with $O_{12} \cap O_{23} = O_{23} \cap O_{31} = \dots = O_{k1} \cap O_{12} = \emptyset$.

For instance a **coalition 3-cycle** is a triple of coalitions C_{12}, C_{23}, C_{31} such that

$$C_{12} = N_1 \cup N_2 \cup O_{12}, \quad C_{23} = N_2 \cup N_3 \cup O_{23}, \quad \text{and} \quad C_{31} = N_3 \cup N_1 \cup O_{31}$$

with N_1, N_2, N_3 and O_{12}, O_{23}, O_{31} satisfying the above conditions. In the roommates profile 3, the coalition 3-cycle corresponds to the situation in which N_1, N_2 and N_3 are singletons, and $O_{12} = O_{23} = O_{31} = \emptyset$.

Let $[C \succ C' \succ \dots]$ represent any preference in which C is ranked first, C' second, and the rest of the ranking is arbitrary. A domain R is **odd-cyclic** if there exists a coalition k-cycle *with k odd* such that

$$\begin{aligned}
&R_i \supseteq \{[C_{j(j+1)} \succ_i^1 C_{(j-1)j} \succ_i^1 \dots], [C_{(j-1)j} \succ_i^2 C_{j(j+1)} \succ_i^2 \dots]\} \\
&\quad \text{for all } i \in N_j \text{ and all } j \in \{2, \dots, k-1\}, \\
&R_i \supseteq \{[C_{12} \succ_i^1 C_{k1} \succ_i^1 \dots], [C_{k1} \succ_i^2 C_{12} \succ_i^2 \dots]\} \\
&\quad \text{for all } i \in N_1, \\
&R_i \supseteq \{[C_{k1} \succ_i^1 C_{(k-1)k} \succ_i^1 \dots], [C_{(k-1)k} \succ_i^2 C_{k1} \succ_i^2 \dots]\} \\
&\quad \text{for all } i \in N_k, \text{ and} \\
&R_i \supseteq \{[C_{jh} \succ_i^* \dots]\} \\
&\quad \text{for all } i \in O_{jh} \text{ and all } O_{jh} \in \{O_{12}, \dots, O_{k1}\}.
\end{aligned} \tag{8}$$

For example, a domain R is **odd-cyclic** if there exists a coalition 3-cycle such that

$$\begin{aligned}
&R_i \supseteq \{[C_{12} \succ_i^1 C_{31} \succ_i^1 \dots], [C_{31} \succ_i^2 C_{12} \succ_i^2 \dots]\} && \text{for all } i \in N_1, \\
&R_i \supseteq \{[C_{23} \succ_i^1 C_{12} \succ_i^1 \dots], [C_{12} \succ_i^2 C_{23} \succ_i^2 \dots]\} && \text{for all } i \in N_2, \\
&R_i \supseteq \{[C_{31} \succ_i^1 C_{23} \succ_i^1 \dots], [C_{23} \succ_i^2 C_{31} \succ_i^2 \dots]\} && \text{for all } i \in N_3, \\
&R_i \supseteq \{[C_{12} \succ_i^* \dots],\} && \text{for all } i \in O_{12}, \\
&R_i \supseteq \{[C_{23} \succ_i^* \dots],\} && \text{for all } i \in O_{23}, \text{ and} \\
&R_i \supseteq \{[C_{31} \succ_i^* \dots],\} && \text{for all } i \in O_{31}.
\end{aligned}$$

In the roommates domain described in the text we have

$$\begin{aligned}
&R_1 \supseteq \{[\{1,2\} \succ_1^1 \{3,1\} \succ_1^1 \dots], [\{3,1\} \succ_1^2 \{1,2\} \succ_1^2 \dots]\} \\
&R_2 \supseteq \{[\{2,3\} \succ_2^1 \{1,2\} \succ_2^1 \dots], [\{1,2\} \succ_2^2 \{2,3\} \succ_2^2 \dots]\}, \text{ and} \\
&R_3 \supseteq \{[\{3,1\} \succ_3^1 \{2,3\} \succ_i^1 \dots], [\{2,3\} \succ_3^2 \{3,1\} \succ_i^2 \dots]\},
\end{aligned}$$

A domain is **individually odd-cyclic** if it is odd-cyclic for some $N_1, \dots, N_k$ with $\#N_1 = \dots = \#N_k = 1$.

Proof of Proposition 2. The proof follows the same logic the argument about profile (3) in the text. The proof is for (i). For (ii), simply replace any joint deviation by players N_i by a deviation from the only $i \in N_i$ (and truthful *strong* Nash equilibrium by truthful Nash equilibrium). In order to derive a contradiction, suppose that M has a truthful strong Nash equilibrium on R and that M is a soulmates mechanism.

Consider profile $\widetilde{\succ}$ in which (a) players in $O_{12}, \ldots, O_{k1}$ have preference $\succ_i^*$, and (b) players in $i \in (N_1 \cup \cdots \cup N_3)$ have preference $\succ_i^1$ (see the definition of an odd cyclic profile above). Because R is an odd cyclic domain, there exists such an $\widetilde{\succ} \in R$. Hence, because M has a truthful strong Nash equilibrium on R, no coalition of players can deviate when players report profile $\widetilde{\succ}$.

This implies that either C_{12} or C_{k1} must form. Otherwise, N_1 can jointly deviate by reporting $\succ_{N_1}^2$, that is pretending they are the soulmates of players in N_k.

Suppose that C_{12} forms. If players in N_3 are matched with their second ranked coalition, they could deviate by reporting $\succ_{N_3}^2$, that is pretending they are the soulmates of players in N_2. To prevent this profitable joint deviation, the players in N_3 must be matched with their best coalition C_{34}.

Extending the argument by induction, all coalitions $C_{12}, C_{34}, \ldots, C_{j(j+1)}$ must form, for any odd $j \leq k$. But because k is odd this implies that neither C_{1k} nor $C_{(k-1)k}$ form (because players in N_1 are already in C_{12} and players in N_{k-1} are already in $C_{(k-2)(k-1)}$). However, players in N_{k-1} are matched with their second ranked coalition, and the players in N_k can therefore jointly deviate by reporting preference $\succ_k^2$ in which they are the soulmates of the players in N_{k-1}.

Similarly, if C_{23} forms, the players in N_1 can jointly deviate by reporting preference $\succ_1^2$ in which they are the soulmates of the players in N_k. In both cases, a coalition of players can deviate when players report $\widetilde{\succ}$, a contradiction. □

C.1 Unconstrained Preferences

Recall that common ranking profiles are those in which if player i prefers to be matched with j over k then every other player prefers to be matched with j over k.[27]

Reciprocal preferences are preferences such that, for any player i, if i prefers coalition T to all other coalitions, then for all $j \in T$, j also prefers coalition T to all other coalitions.[28]

[27] Common-ranking profiles are IMS-complete since they satisfy the top coalition property. For any subset of the players, two most highly ranked players form a top coalition.

[28] Reciprocal preferences are always IMS-complete since reciprocity partitions the set of players into coalitions of soulmates. Because reciprocity requires no structure on preferences other than each player's top choice of coalition, reciprocal preferences need not have the top-coalition property. Furthermore, the top-coalition property does not require that every player is in a top coalition within the entire set of players but only that there is at least *one* top coalition.

Also, common-ranking and reciprocal preferences are incompatible since common-ranking

To do our analysis, we randomly generated $10,000$ preference profiles for total player set sizes $n = 4, 6, 8$, and 10 and tested the IMS-completeness and the top-coalition property. We have limited the size of player sets since it is computationally hard to test every subset of players to determine whether even a particular profile has the top-coalition property. Also, even the simple problem of counting preference profiles that contain *at least some* soulmates is a complex problem, not conducive to standard counting techniques. For some analytical results on this problem see Supplementary materials, Section D.

The proportion of common-ranking and reciprocal roommates profiles are much easier to count than IMS-complete and top-coalition profiles. The number of common-ranking profiles is $n!$. Hence, the proportion $\frac{n!}{(n-1)!^n}$ of common-ranking profiles is very small. Even for total player sets of size 6 the proportion is 2.41×10^{-10}. There are $\frac{n!((n-2)!)^n}{2^{\frac{n}{2}}}$ reciprocal profiles. In comparison to common-ranking, reciprocal profiles are far more abundant. When $n = 6$ about a half of one percent of all profiles are reciprocal. However, each of these accounts for only a small portion of the IMS-complete profiles.

Table 2 compares the approximate proportions of IMS-complete, top-coalition, common-ranking, and reciprocal profiles. Top-coalition profiles are only a small portion of IMS-complete profiles as well. In fact, when the number of players is greater than or equal to 8, there were no top-coalition profiles among the 10,000 test cases.

	IMS-complete	Top-Coalition	Common-Ranking	Reciprocal
n=4	0.6249[29]	0.3442	0.0185	0.0741
6	0.3064	0.0087	0.0000	0.0057
8	0.1219	0.0000	0.0000	0.0004
10	0.0460	0.0000	0.0000	0.0000

Table 2: Approximate proportions of unconstrained profiles meeting each of the conditions from 10,000 randomly generated test-cases.

Even though IMS-complete profiles are far more abundant, they still form a vanishing subset of profiles under unconstrained preferences. In the next two subsections, we consider preference domains endowed with natural structure.

C.2 Soulmates Under Relaxed Reciprocity

Relaxed forms of reciprocity are natural in some environments: if Alice prefers Alex to all others, it may be that Alex ranks Alice highly as well. If preferences are purely reciprocal then the preference profile is IMS-complete, but what happens under relaxed reciprocity?

requires that everyone has the same favorite partners while reciprocal preferences require players have unique favorite partners.

[29] This proportion can be calculated analytically using the results in Supplementary materials, Section D. The true value is $\frac{816}{1296} \approx 0.6296$.

To study IMS under different degrees of reciprocity, we now introduce a generalization of reciprocal preferences, *k-reciprocal* preferences. For the roommates problem, a profile $\succ$ is **k-reciprocal** if for any two players i and j, j is in i's top k most preferred players *if and only if* i is in j's top k most preferred players.

Example 6. Consider the following profile of preferences over partners in a roommates profile.

$$\begin{array}{lccccc} 1: & 3 & \succ_1 & 2 & \succ_1 & 4 \\ 2: & 1 & \succ_2 & 4 & \succ_2 & 3 \\ 3: & 4 & \succ_3 & 1 & \succ_3 & 2 \\ 4: & 2 & \succ_4 & 3 & \succ_4 & 1 \end{array}$$

The profile is trivially 3-reciprocal, as there are only 3 possible partners for each player. In general, $n-1$ reciprocal preferences are unconstrained preferences.

The profile is also 2-reciprocal. For example, player 1's two most preferred partners are players 3 and 2, while 1 is the most preferred partner for player 2 and the second most preferred for player 3.

As player 3 is player 1's most preferred partner and 1 is not 3's most preferred partner, the profile is not 1-reciprocal.

We randomly generated $10,000$ k-reciprocal profiles for each combination of total number of players $n \in \{4, 6, 8, 10\}$ and $k \in \{2, 4, 6, n-1\}$ (recall that $k = n-1$ are unconstrained profiles), tested each profile for IMS-completeness and recorded the number of players matched by IMS. Results for $N = 10$ are show in figure 4 and full results are reported in Supplementary materials, table 4.

Most 2-*reciprocal* profiles are IMS-complete. While, as expected, the percentage of IMS-complete profiles decreases k, a large portion of players can be matched by IMS even with moderate amounts of commonality. For instance, on average in groups of ten, about half of players can be matched by IMS. For comparison, on average only about 16% can be matched in unconstrained profiles. Interestingly, for $k =$ 2,4, and 6, the average proportion of players that can be matched by IMS is nearly constant as n increases.

C.3 Soulmates Under Relaxed Common-Ranking

We now consider a relaxed form of common-ranking where players' preferences are partially correlated. Unlike in the previous section, where we study a class of preference profiles, here we will instead study distributions over preference profiles that can be thought of as "noisy" common rankings.

In this experiment, we randomly generated 10,000 noisy common-ranking profiles for $N = 6, 8, 10$ using the following procedure. Each player's preference starts with an underlying common cardinal vector of utilities over partners. Player's utilities are then perturbed via a normally distributed "noise" term. A player's new ordinal ranking becomes the player's preference. The larger

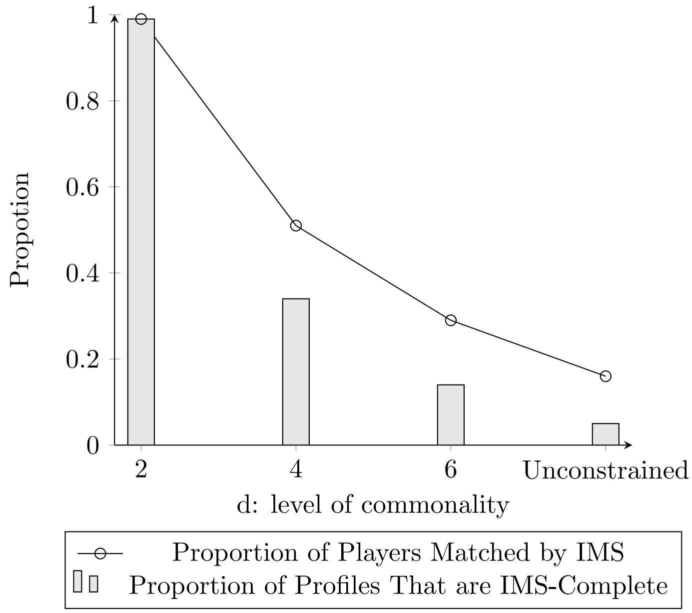

Figure 4: Proportion of Players Matched and IMS-Complete Profiles by Level of Commonality (D) for $N = 10$

the variance of the noise, the larger the deviation of the expected perturbed preference from the original common-ranking profile.

To make the variance of the noise term informative of the extent of commonality of rankings, we calibrated variances to target specific amounts of ordinal distance between the rankings. We measure the distance between any two players' ordinal rankings using the Kemeny distance (swap-distance, Kemeny (1959)). The Kemeny distance counts the number of pairwise components which have a different ordering in two preference lists.

In the experiment, we chose noise amounts that correspond to an average Kemeny distance between any two players ranging from 0.2 to 2 (in 0.2 increments). For instance, $d = 1$ indicates that, on average, it takes a "swap" of a single pairwise preference component to transform one player's preference list into another.[30]

Figure 5 below reports the results for $N = 10$ and the full results are reported in Supplementary materials, table 5. Even with a moderate commonality, a large proportion of profiles are IMS-complete. For example, with $N = 10$, approximately two-thirds of profiles are IMS-complete when the variance in common ranking is such that player's preference lists difference by on-average one swap ($d = 1$). Approximately one-half of profiles are IMS-complete for

[30]For example, if the two preferences are $1 \succ 2 \succ 3$ and $2 \succ' 1 \succ' 3$, then the Kemeny distance is one, as swapping player 2 for player 1 in one of the preferences makes the two preferences identical.

$d = 2$. Interestingly, the likelihood of a profile being IMS-complete does not decrease much with the size of the total player set, holding the average distance constant.

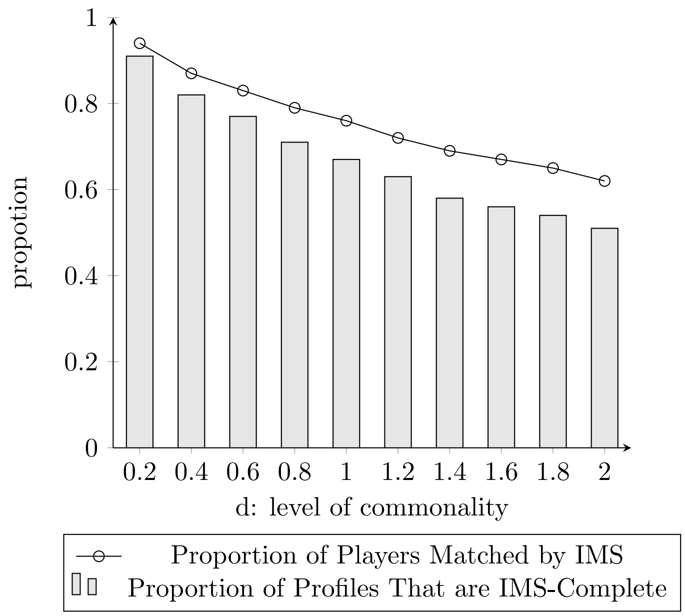

Figure 5: Proportion of Players Matched and IMS-Complete Profiles for by Level of Commonality (D) for $N = 10$

D Analytic Results on Counting IMS-complete Profiles

In the roommates problem, the players' favorite partners can be represented by an endofunction $f : N \to N$ or it's equivalent functional graph (directed 1-forest). A pair of soulmates is a fixed point of $f(f(i))$ or alternatively a 2-cycle of it's functional graph.

These objects are too complex to be directly counted using standard methods. However, Fripertinger and Schöpf (1999) provides useful results by using Polya enumeration theory. Their corollary 6 provides a count of functional graphs without 1 or 2-cycles. Dividing this count by the number of functional graphs with no 1-cycles $(n-1)^n$ provides the proportion of preference profiles (of those where being matched is better than being alone) for n players in which there are no soulmates. Subtracting this from 1 provides $N(n)$, the probability that at least some players can be matched during IMS:

$$N(n) = 1 - \frac{1}{(n-1)^n} \sum_{j=0}^{n-1} j! \binom{n-s}{j} n^{n-s-j} Z \left(S_j, j | x_k = \begin{cases} -1 & k \in L \\ 0 & k \notin L \end{cases} \right)$$

Where Z is the symmetric group cycle index polynomial.

It is possible to use $N(n)$ to find an upper bound on the probability that IMS matches everyone when preferences are uniform by using the approximation that the preferences players have over the remaining players when a pair of soulmates are removed is also uniform. This overestimates the probability that there will be a set of soulmates in the remaining players since the fact that a particular player was not just matched with a soulmate implies it is more likely that her favorite among the remaining players does not also like her best.

Using this bound however, the approximate probability that there are more soulmates to match after removing the first pair is $N(n-2)$. Continuing this, the approximate probability that IMS-completes is the probability that there continue to be "more soulmates to remove" as the group dwindles from N to 2. This is the product:

$$N(n) \cdot N(n-2) \cdot N(n-4) \cdots N(2)$$

This yields the following bounds,[31] which are compared to the computed proportions from Section 6

	Upper Bound	Computed
n=4	0.6296	0.6249
6	0.3330	0.3064
8	0.1624	0.1219
10	0.0755	0.0460

Table 3: Bound and Computed Proportion of IMS-complete Profiles.

E Computational Tables

[31] For $n = 4$ this procedure yields the true proportion since there are always soulmates when $n = 2$.

	Proportion of Players Matched				IMS-complete Proportion			
	r=2	r=4	r=6	r=n-1	r=2	r=4	r=6	r=n-1
N=8	0.99	0.51	0.30	0.24	0.99	0.39	0.18	0.12
10	0.99	0.51	0.29	0.16	0.99	0.34	0.14	0.05
12	0.99	0.50	0.29	0.12	0.99	0.28	0.11	0.01
14	0.99	0.49	0.29	0.09	0.99	0.23	0.08	0.00
16	0.99	0.49	0.28	0.08	0.97	0.17	0.06	0.00
18	0.99	0.49	0.28	0.07	0.94	0.15	0.04	0.00
20	0.98	0.49	0.27	0.06	0.92	0.12	0.03	0.00

Table 4: Proportion of Players Matched and IMS-complete Profiles by level of reciprocity (r) and group size (N).

	Proportion of Players Matched			IMS-complete Proportion		
	N=6	N=8	N=10	N=6	N=8	N=10
k=0.2	0.93	0.93	0.94	0.91	0.91	0.91
0.4	0.86	0.88	0.87	0.84	0.84	0.82
0.6	0.82	0.82	0.83	0.78	0.77	0.77
0.8	0.77	0.78	0.79	0.73	0.72	0.71
1.0	0.73	0.74	0.76	0.69	0.67	0.67
1.2	0.70	0.72	0.72	0.65	0.64	0.63
1.4	0.67	0.69	0.69	0.62	0.61	0.58
1.6	0.65	0.66	0.67	0.60	0.58	0.56
1.8	0.62	0.64	0.65	0.57	0.55	0.54
2.0	0.59	0.62	0.62	0.54	0.53	0.51

Table 5: Proportion of Players Matched and IMS-complete Profiles by group size (N) and level of commonality (k).