0.1 Cournot Model of Oligopoly.

Inverse demand is $p(Q) = 25 - Q$, and the cost function of each firm is $c(q_i) = q_i^2$. Suppose there are two firms.

$$\pi_1(q_1, q_2) = q_1(25 - (q_1 + q_2)) - q_1^2$$

$$\pi_2(q_1, q_2) = q_2(25 - (q_1 + q_2)) - q_2^2$$

Both maximize profit:

$$\frac{\partial \left(q_1(25 - (q_1 + q_2)) - q_1^2\right)}{\partial q_1} = 0$$

$$\frac{\partial \left(q_2(25 - (q_1 + q_2)) - q_2^2\right)}{\partial q_2} = 0$$

The best response functions are each firm's optimal quantity given the other firm's quantity. Here are the two first order conditions:

$$-4q_1 - q_2 + 25 = 0$$

$$-q_1 - 4q_2 + 25 = 0$$

Solve these respectively for q_1 and q_2 to get the best response functions:

$$q_1 = \frac{1}{4}(25 - q_2)$$

$$q_2 = \frac{1}{4}(25 - q_1)$$

These are the best responses. An equilibrium occurs at a (q_1, q_2)such that q_1 is a best response to q_2 and q_2 is a best response to q_1. "Mutually best responses" give us a **Nash Equilibrium.** Normally we would solve these simultaenously to find the equilibrium.

$$q_1 = \frac{1}{4}(25 - q_2)$$

$$q_2 = \frac{1}{4}(25 - q_1)$$

Since the firms are symmetric, we can implose symmetry on any best response and solve it to find a "symmetric" equilibrium.

$$q = \frac{1}{4}(25 - q)$$

$$q = 5$$

The nash equilibrium in this game is $(5, 5)$.

Let's look at the profit of a firm. Firm 1. Suppose firm 2 is setting $q_2 = 5$

$$\pi(q_1, 5) = q_1(25 - (q_1 + 5)) - q_1^2$$

$$= 2(10 - q_1)q_1$$

This is a parabola that is maximized at 5.

Let's calculate the profit for each firm at the equilibrium:

$$\pi(5, 5) = 5(25 - (5 + 5)) - 5^2$$

$$\pi(5, 5) = 50$$

1 Collusion and Cooperation

Suppose the firms get together and decide to ignore their competition with each-other and work together to maximize their profits. **Collusion.**

Suppose the firms agree to each pick the same quantity q that maximizes their joint profits.

$$\left(q_1(25 - (q_1 + q_2)) - q_1^2\right) + \left(q_2(25 - (q_1 + q_2)) - q_2^2\right)$$

Both firms agree to set the same quantity q

$$= 2\left(q(25 - (q + q)) - q^2\right)$$

This is the total profit of the two firms conditional on them both choosing quantity q. What is the q that maximizes that profit?

In nash equilibrium, both choose 5. Both earned 50. Total profit is 100.

Solve:

$$\frac{\partial\left(2\left(q\left(25-(q+q)\right)-q^2\right)\right)}{\partial q}=0$$

$$50-12q=0$$

$$q=\frac{50}{12}=4.16667$$

How much do they earn?

$$\pi\left(\frac{50}{12},\frac{50}{12}\right)=\frac{50}{12}\left(25-\left(\frac{50}{12}+\frac{50}{12}\right)\right)-\frac{50}{12}^2=52.1$$

This is not an equilibrium. Either firm would have incentive to undermine the agreement and earn even more!

Suppose firm 1 believes firm 2 will go along with the agreement. What is firm 1's best response?

$$q_1=\frac{1}{4}\left(25-q_2\right)$$

Find firm 1's best response to $\frac{50}{12}$

$$q_1=\frac{1}{4}\left(25-\frac{50}{12}\right)$$

$$q_1=\frac{125}{24}=5.20833$$

By deviating from the agreement, firm 1 earns:

$$\pi_1\left(\frac{125}{24},\frac{50}{12}\right)=\frac{125}{24}\left(25-\left(\frac{125}{24}+\frac{50}{12}\right)\right)-\left(\frac{125}{24}\right)^2=54.3$$

Firm 2 earns:

$$\pi_1\left(\frac{125}{24},\frac{50}{12}\right)=\frac{50}{12}\left(25-\left(\frac{125}{24}+\frac{50}{12}\right)\right)-\left(\frac{50}{12}\right)^2=48.8$$

1.1 Prisoner's Dilemma of Collusion

A **Normal Form** $2x2$ **game.** These are the simplest game theory games.

Players, Strategies, Payoffs.

	cooperate $q_2 = \frac{25}{6}$	deviate $q_2 = \frac{125}{24}$
cooperate $q_1 = \frac{25}{6}$	$52.1, 52.1$	$47.7, \mathbf{54.3}$
deviate $q_1 = \frac{125}{24}$	$\mathbf{54.3}, 47.7$	$\mathbf{48.8}, \mathbf{48.8}$

Figure 1: The prisoner's dilemma as applied to Cournot Collusion.

1.2 Prisoner's Dilemma

	cooperate	deviate
cooperate	4,4	0,**10**
deviate	**10**,0	**2**,**2**

Figure 2: A simplified prisoner's dilemma game.

Cooperation is not an equilibrium in the prisoner's dilemma. Both deviating is the only equilibrium.

This is a game called a "social dilemma".

1.3 Coordination Game

	a	b
a	4,4	0,1
b	1,0	3,3

Figure 3: A coordination game.

What is the nash equilibrium of this game?

	a	b
a	**4,4**	0,1
b	1,0	**3,3**

Figure 4: A coordination game.

This game has two nash equilibrium (is pure strategies). (a, a) or (b, b).

Notice in this game our interests are aligned.

1.4 Sustaining Cooperation In Repeated Interactions.

How can we possibly sustain non-equilibrium outcomes in game theory? How can we sustain cooperation in the prisoner's dilemma.

Let's suppose the players in a prisoner's dilemma play the game every day.

	cooperate	deviate
cooperate	4,4	0,10
deviate	10,0	2,2

Figure 5: A simplified prisoner's dilemma game.

And their utility is a discounted stream of payments. Each player cares about tomorrow β times how much they care about today.

Suppose the players play the nash equilibrium every day. Each player's utility:

$$2 + \beta (2) + \beta^2 (2) + \beta^3 (2) + ... = \sum_{t=0}^{\infty} \beta^t (2)$$

$$2 \sum_{t=0}^{\infty} \beta^t = 2 \left(\frac{1}{1-\beta} \right)$$

My utility for playing the nash equilibrium forever. Suppose $\beta = 0.5$

$$2 + 1 + 0.5 + 0.25 + 0.125 + ...$$

$$2 \left(\frac{1}{1-0.5} \right) = 2 \frac{1}{0.5} = 2 * 2 = 4$$