1 Utility

1.1 Definition

Utility is a function that maps the consumption set / fesible set X into numbers.

$$U(1,1) = 2$$

$$U(2,1) = 3$$

$$U(5,7) = 12$$

A utility function $U(x)$ represents preferences $\succsim$ when, for every pair of bundles x and y, $U(x) \geq U(y)$ if and only if $x \succsim y$. Informally, better bundles get higher numbers.

1.2 Transformations

Any strictly increasing function (monotonic transformation) of a utility function represents the same preferences as the original utility function.

$U(x_1, x_2) = x_1 + x_2$

Here are some utility functions that represent the same preferences.

$(x_1 + x_2)^2$

$(x_1 + x_2) + 2$

$ln(x_1 + x_2)$

$\sqrt{(x_1 + x_2)}$

There are many ways to represent the same preferences, and since we only care about representing the underlying preferences, all are equally valid utility functions.

Here is another example:

$u(x_1, x_2) = x_1^\alpha x_2^\beta$

$$ln\left(x_1^\alpha x_2^\beta\right) = ln(x_1^\alpha) + ln\left(x_2^\beta\right) = \alpha ln(x_1) + \beta ln(x_2)$$

Thus, $\alpha ln(x_1) + \beta ln(x_2)$ and $x_1^\alpha x_2^\beta$ represent the same preferences. The former sometimes easier to work with.

1.3 MRS from Utility Function

Recall, MRS (marginal rate of substitution) is the slope of the indifference curve. It measures, roughly, how much x_2 will you give up to get one more x_1.

$$MRS = -\frac{\frac{\partial(u(x_1,x_2))}{\partial x_1}}{\frac{\partial(u(x_1,x_2))}{\partial x_2}} = -\frac{MU_1}{MU_2}$$

MU_1 and MU_2 are the "marignal utilities" of the utility function. They measure how does utility increase when I increase that good by a little bit.
Suppose $MU_1 = 2$ and $MU_2 = 1$.

$$MRS = -\frac{2}{1}$$

MRS for $x_1 + x_2$

$$-\frac{\frac{\partial(x_1+x_2)}{\partial x_1}}{\frac{\partial(x_1+x_2)}{\partial x_2}} = -\frac{1}{1}$$

Willing to give up 1 unit of good 2 to get one unit of good 1.
MRS for $(x_1 + x_2)^2$

$$-\frac{\frac{\partial\left((x_1+x_2)^2\right)}{\partial x_1}}{\frac{\partial\left((x_1+x_2)^2\right)}{\partial x_2}} = -\frac{2\,(x_1 + x_2) * 1}{2\,(x_1 + x_2) * 1} = -1$$

MRS for $ln\,(x_1 + x_2)$

$$-\frac{\frac{\partial(ln(x_1+x_2))}{\partial x_1}}{\frac{\partial(ln(x_1+x_2))}{\partial x_2}} = -\frac{\frac{1}{x_1+x_2}1}{\frac{1}{x_1+x_2}1} = -1$$

If the MRS is the same for two utility functions, they represent the same preferences.
MRS for $x_1^2 x_2^2$

$$-\frac{\frac{\partial\left(x_1^2 x_2^2\right)}{\partial x_1}}{\frac{\partial\left(x_1^2 x_2^2\right)}{\partial x_2}} = -\frac{2x_1 x_2^2}{2x_1^2 x_2} = -\frac{x_2}{x_1}$$

MRS $ln\,(x_1) + x_2$
$MU_1 = \frac{\partial(ln(x_1)+x_2)}{\partial(x_1)} = \frac{1}{x_1}$
$MU_2 = \frac{\partial(ln(x_1)+x_2)}{\partial(x_2)} = 1$

$$-\frac{\frac{\partial(ln(x_1)+x_2)}{\partial(x_1)}}{\frac{\partial(ln(x_1)+x_2)}{\partial(x_2)}} = -\frac{\frac{1}{x_1}}{1} = -\frac{1}{x_1}$$

1.4 Examples of Utility Functions

Canonical forms of utility functions.

1.4.1 Perfect Substitutes

$u(x_1, x_2) = ax_1 + bx_2$
Finn's preferences for ice cream. Only cares about number of scoops: $x_1 + x_2$
Preferences over eggs and dozens of eggs (x_2). $u(x_1, x_2) = x_1 + 12x_2$
For $u(x_1, x_2) = ax_1 + bx_2$ $MRS = -\frac{a}{b}$

1.4.2 Quasi-Linear

$u(x_1, x_2) = f(x_1) + x_2$
Examples:
$u(x_1, x_2) = ln(x_1) + x_2$. $MRS = -\frac{1}{x_1}$
$u(x_1, x_2) = \sqrt{x_1} + x_2$. $MRS = -\frac{\frac{\partial(\sqrt{x_1}+x_2)}{\partial x_1}}{\frac{\partial(\sqrt{x_1}+x_2)}{\partial x_2}} = -\frac{\frac{\partial\left(x_1^{\frac{1}{2}}+x_2\right)}{\partial x_1}}{\frac{\partial\left(x_1^{\frac{1}{2}}+x_2\right)}{\partial x_2}} = -\frac{\frac{1}{2}x_1^{-\frac{1}{2}}}{1} = -\frac{1}{2}\frac{1}{x_1^{\frac{1}{2}}}$

1.4.3 Cobb-Douglas

$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$

$$MRS = -\frac{\frac{\partial\left(x_1^{\alpha}x_2^{\beta}\right)}{\partial(x_1)}}{\frac{\partial\left(x_1^{\alpha}x_2^{\beta}\right)}{\partial(x_2)}} = -\frac{\alpha x_2}{\beta x_1} = -\frac{\alpha}{\beta}\frac{x_2}{x_1}$$

1.4.4 Perfect Complements

Left and right shoes $u(x_1, x_2) = min\{x_1, x_2\}$
$min\{10, 20\} = 10$
Pies?
2 apples (x_1) and 1 crust (x_2).
$(4, 1)$
Most pies I can make with 4 apples is 2. $\frac{x_1}{2}$
Most pies I can make with 1 crust is 1. x_2

$$min\left\{\frac{x_1}{2}, x_2\right\}$$

To get the "no waste line" set the two things inside the min equal to eachother.

$$\frac{x_1}{2} = x_2$$

3 widgets (x_1) and 2 hours of labor (x_2) to make a bicycle.

$$min\left\{\frac{x_1}{3}, \frac{x_2}{2}\right\}$$

Line of no waste:

$$\frac{x_1}{3} = \frac{x_2}{2}$$

$$\frac{2}{3}x_1 = x_2$$

1.5 Utility to Indifference Curves

Tricks for drawing indifference curves.

$$u\left(x_1, x_2\right) = x_1 + 2x_2$$

Let's find the indifference curve that gives utility 10. All of the combinations of x_1 and x_2 that give utility of 10 are:

$$x_1 + 2x_2 = 10$$

$$2x_2 = 10 - x_1$$

$$x_2 = 5 - \frac{1}{2}x_1$$

Let's find the indifference curve that gives utility 20. All of the combinations of x_1 and x_2 that give utility of 20 are:

$$x_1 + 2x_2 = 20$$

$$x_2 = 10 - \frac{1}{2}x_1$$

Let's look at the indifference curves for $ln\left(x_1\right) + x_2$

Indifference curve for utility of 5:

$$ln\left(x_1\right) + x_2 = 5$$

$$x_2 = 5 - ln\left(x_1\right)$$

$x_1 = 1$?

$$x_2 = 5 - ln\left(1.0\right) = 5.$$

$(1, 5)$ is on this indifference curve
$x_1 = 10$

$$x_2 = 5 - ln\left(10\right) = 2.69741$$

$$(10, 2.69741)$$