1 Cost Min / Profit Max

The firm figures out the cheapest way to produce any output amount y.
Construct the cost function $c(y)$.
Construct the profit function in terms of y.

$$\pi(y) = p(y)\,y - c(y)$$

$$\frac{\partial(\pi(y))}{\partial y} = \frac{\partial(p(y)\,y)}{\partial y} - \frac{\partial(c(y))}{\partial y} = mr - mc$$

$$mr - mc = 0$$

$$mr = mc$$

If this wasn't true then either $mr > mc$ in which case increasing output would increase revenue faster than it would increase cost. In this case, I should increase output because revenue will go up faster than cost and thus increase profit.
If this wasn't true then either $mr < mc$ in which case decreasing output would decreases cost faster than it would decrease revenue. In this case, I should decrease output.

If we make the price-taking assumption (perfect competition) then $p(y)$ is fixed at p.
Under price-taking, the firm assumes that the price they get per unit does not depend on their output.

$$\pi(y) = py - c(y)$$

This is maximized where the marginal profit is zero.

$$p = mc$$

1.1 Cobb Douglass

$f(x_1, x_1) = x_1 x_2$. $w_1 = 1, w_2 = 1$.
Look for where the TRS is equal to the negative of the ratio of prices

$$-\frac{x_2}{x_1} = -\frac{1}{1}$$

Two constraints (equal-slope and production):

$$x_2 = x_1$$

$$x_1 x_2 = y$$

Solve these:

$$x_1^2 = y$$

Conditional factor demands:

$$x_1 = \sqrt{y}$$

$$x_2 = \sqrt{y}$$

Note that if they use the input bundle $\left(\sqrt{y}, \sqrt{y}\right)$ then output is $f\left(\sqrt{y}, \sqrt{y}\right) = \sqrt{y}\sqrt{y} = y$

To find the cost function, plug the conditional factor demands into:

$$w_1 x_1 + w_2 x_2$$

$$1\sqrt{y} + 1\sqrt{y} = 2\sqrt{y}$$

$$c(y) = 2\sqrt{y}$$

$$c(1) = 2$$

$$c(4) = 2\sqrt{4} = 4$$

Because this firm has increasing returns to scale, it becomes more and more efficient as it uses more and more inputs. This is reflected in the fact that the costs are increasing less and less as we produce more.

$$\frac{\partial\left(2\sqrt{y}\right)}{\partial y} = \frac{1}{\sqrt{y}}$$

Decreasing marginal cost: each additional unit of output costs less than the one before it.

1.2 Profit max for the previous example.

$p = 4$

$$\pi(y) = 4y - 2\sqrt{y}$$

$$4y - 2\sqrt{y}$$

Recall that for a firm in perfect competition, if they have increasing or linear returns to scale, if there is ever an output that gives them positive profit, there is no profit maximizing level of output.

1.3 Cobb Douglass

$x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$, $w_1 = 1, w_2 = 1$

$$-\frac{x_2}{x_1} = -\frac{1}{1}$$

$$x_1 = x_2$$

$$x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} = y$$

Solve these together:

$$x_1^{\frac{1}{3}} x_1^{\frac{1}{3}} = y$$

$$x_1^{\frac{2}{3}} = y$$

$$x_1 = y^{\frac{3}{2}}$$

$$x_2 = y^{\frac{3}{2}}$$

Cost function:

$$c(y) = 1y^{\frac{3}{2}} + 1y^{\frac{3}{2}} = 2y^{\frac{3}{2}}$$

$$c(y) = 2y^{\frac{3}{2}}$$

Profit function. $p = 4$

$$\pi(y) = 4y - 2y^{\frac{3}{2}}$$

$$\frac{\partial\left(4y - 2y^{\frac{3}{2}}\right)}{\partial y} = 4 - \left(\frac{3}{2}2y^{\frac{3}{2}-1}\right) = 4 - 3y^{\frac{1}{2}}$$

$$4 - 3y^{\frac{1}{2}} = 0$$

$$\frac{4}{3} = y^{\frac{1}{2}}$$

$$\left(\frac{4}{3}\right)^2 = y$$

$$\frac{16}{9} = y$$

1.4 Min Function

$(min\{x_1, x_2\})$
No waste condition:

$$x_1 = x_2$$

Production constraint:

$$min\{x_1, x_2\} = y$$

Solve these:

$$min\{x_1, x_1\} = y$$

$$x_1 = y$$

$$x_2 = y$$

Cost function (assume $w_1 = w_2 = 1$)

$$y + y = 2y$$

$$c(y) = 2y$$

Suppose $p = 4$

$$\pi(y) = 4y - 2y = 2y$$

No profit maximizing level of output.

1.5 Min Function

$(min\{x_1, x_2\})^{\frac{1}{2}}$

$$x_1 = x_2$$

$$(min\{x_1, x_2\})^{\frac{1}{2}} = y$$

Solve these:

$$(min\{x_1, x_1\})^{\frac{1}{2}} = y$$

$$(x_1)^{\frac{1}{2}} = y$$

$$x_1 = y^2$$

$$x_2 = y^2$$

$w_1 = w_2 = 1$

$$c(y) = 2y^2$$

Assuming $p = 4$

$$\pi(y) = 4y - 2y^2$$

$$\frac{\partial\left(4y - 2y^2\right)}{\partial y} = 4 - 4y$$

$$4 - 4y = 0$$

$$4 = 4y$$

$$y = 1$$