0.1 Tragedy of the Commons Example

Suppose $100\sqrt{B}$ fish will be caught on a lake when B boats are fishing. Fish can be sold for $\$1$ but it costs $\$10$ to buy fuel and supplies to fish.

Fish caught per boat.

$$\frac{100\sqrt{B}}{B}$$

Profit of each boat:

$$\frac{100\sqrt{B}}{B} - 10$$

$$\pi\left(B\right) = \frac{100}{\sqrt{B}} - 10$$

$$\pi\left(1\right) = \frac{100}{1} - 10 = 90$$

$$\pi\left(2\right) = \frac{100}{\sqrt{2}} - 10 = 60.7107$$

$$\pi\left(3\right) = \frac{100}{\sqrt{3}} - 10.0 = 47.735$$

To find the individually-optimal entry of boats, we calculate the B that makes profit zero for all boats.

$$\frac{100}{\sqrt{B}} - 10 = 0$$

$$\frac{100}{\sqrt{B}} = 10$$

$$10 = \sqrt{B}$$

$$B = 100$$

$$\pi\left(100\right) = \frac{100}{\sqrt{100}} - 10 = \frac{100}{10} - 10 = 0$$

$$\pi\left(101\right) = -0.0496281$$

0.1.1 Socially Optimal

The level of entry that maximizes the joint profit of the boats:

$$\left(100\sqrt{B}\right) - 10B$$

We take the derivative with respect to B and find where marginal profit is zero:

$$\frac{\partial\left(\left(100\sqrt{B}\right) - 10B\right)}{\partial B} = \frac{50}{\sqrt{B}} - 10$$

$$\frac{50}{\sqrt{B}} - 10 = 0$$

$$5 = \sqrt{B}$$

$$B = 25$$

$$\frac{\partial\left(100\sqrt{B}\right)}{\partial B} = \frac{50}{\sqrt{B}} = 10$$

When 25 boats enter. Total profit is:

$$\pi = 250$$

0.2 Optimal Fishing Fee

How can we get the boats to 25?

Let's have the government impose a fee for fishing f. We want to choose f such that the individually optimal number of boats is 25.

$$\frac{100\sqrt{B}}{B} - 10 - f$$

$$\frac{100}{\sqrt{B}} - 10 - f$$

What does f need to be so that when $B = 25$ profit is zero for all boats.

$$\frac{100}{\sqrt{25}} - 10 - f = 0$$

$$\frac{100}{5} - 10 - f = 0$$

$$20 - 10 - f = 0$$

$$10 - f = 0$$

$$f = 10$$

If the government imposes a fishing fee of 10 then 25 boats will enter.
The government earns $10 * 25 = 250$ revenue from the lake and all boats earn zero profit.

0.3 Positive Externalities - Public Goods

Suppose 100 people share a park and are asked to donate money to it. (A positive externality since everyone shares the park).
Each individual has income $m = 1000000$ and their contribution is g_i.
The total contributions are $G = \sum_{i=1}^{100} g_i$. And for convenience, let $G_{-i} = G - g_i$.
$u(g_i, G_{-i}) = (1000000 - g_i) + 100\sqrt{G} = (1000000 - g_i) + 100\sqrt{g_i + G_{-i}}$

What contribution maximizes each individuals utility?

$$(1000000 - g_i) + 100\sqrt{g_i + G_{-i}}$$

$$\frac{\partial\left((1000000 - g_i) + 100\sqrt{g_i + G_{-i}}\right)}{\partial g_i}$$

$$\frac{\partial\left(1000000 - g_i + 100\left(g_i + G_{-i}\right)^{\frac{1}{2}}\right)}{\partial g_i} = -1 + 100\frac{1}{2}\left(g_i + G_{-i}\right)^{-\frac{1}{2}}$$

$$= \frac{50}{\sqrt{g_i + G_{-i}}} - 1$$

To maximize utility find where the marginal utility is zero:

$$\frac{50}{\sqrt{g_i + G_{-i}}} - 1 = 0$$

$$\frac{50}{\sqrt{g_i + G_{-i}}} = 1$$

$$50 = \sqrt{g_i + G_{-i}}$$

Best response function:

$$2500 - G_{-i} = g_i$$

$$2500 = g_i + G_{-i}$$

$$2500 = G$$

In any equilibrium, the total contributions are 2500. What is the symmetric equilibirum of this game. A g such that if everyone contributes g, it is a Nash equilibrium:

$$2500 - (99g) = g$$

$$2500 = 100g$$

$$\frac{2500}{100} = g$$

$$25 = g$$

0.3.1 Social Optimum

The social optimum is a g such that if everyone contributes g, the utility of each person is maximized.

$$u\left(g_i, G_{-i}\right) = (1000000 - g_i) + 100\sqrt{g_i + G_{-i}}$$

$$u\left(g, 99g\right) = (1000000 - g) + 100\sqrt{100g}$$

$$\frac{\partial\left((1000000-g)+100\sqrt{100g}\right)}{\partial g} = \frac{500}{\sqrt{g}}-1$$

$$\frac{500}{\sqrt{g}}-1=0$$

$$\frac{500}{\sqrt{g}}=1$$

$$\sqrt{g}=500$$

$$g=250000$$

Utility of each individual when the government imposes a contribution of 250000:

$$u\left(250000, 99\left(250000\right)\right) = \left(1000000-250000\right)+100\sqrt{100\left(250000\right)}$$

$$=1250000$$

When individuals decide how much to contribute they all choose 25:

$$u\left(25, 99\left(25\right)\right) = \left(1000000-250000\right)+100\sqrt{100\left(250000\right)}=1004975$$