0.1 Example: Properties of Cobb Douglass

$$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$$

Diminishing marginal product?

$$MP_1 = \frac{\partial \left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} \right)}{\partial x_1} = \frac{1}{3} \frac{x_2^{\frac{1}{3}}}{x_1^{\frac{2}{3}}}$$

$$\frac{1}{3} \frac{x_2^{\frac{1}{3}}}{x_1^{\frac{2}{3}}}$$

$$\frac{\partial \left(\frac{1}{3} \frac{x_2^{\frac{1}{3}}}{x_1^{\frac{2}{3}}} \right)}{\partial x_1} = -\frac{2\sqrt[3]{x_2}}{9x_1^{5/3}}$$

$$-\frac{2\sqrt[3]{x_2}}{9x_1^{5/3}} < 0?$$

This production function has diminishing marginal product.
A cobb douglass production function $x_1^{\alpha} x_2^{\beta}$ is monotonic if $\alpha, \beta > 0$. x_1 has diminishing marginal product if $\alpha < 1$ and x_2 has diminishing marginal product if $\beta < 1$.

$$MP_1 = \frac{\partial \left(x_1^{\alpha} x_2^{\beta} \right)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta}$$

$$\frac{\partial \left(\alpha * x_1^{\alpha-1} x_2^{\beta} \right)}{\partial x_1} = (\alpha - 1)\alpha x_1^{\alpha-2} x_2^{\beta}$$

$$(\alpha - 1)\alpha x_1^{\alpha-2} x_2^{\beta} < 0$$

$$(\alpha - 1)\alpha < 0$$

$$\alpha < 1$$

Monotonic production function that has diminishing marginal product for both inputs, the exponents have to be between 0 and 1.
When does this have decreasing returns to scale?
For $t > 1$

$$f(tx_1, tx_2) < tf(x_1, x_2)$$

$$(tx_1)^{\frac{1}{3}} (tx_2)^{\frac{1}{3}} < t\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right)$$

$$t^{\frac{1}{3}} (x_1)^{\frac{1}{3}} t^{\frac{1}{3}} (x_2)^{\frac{1}{3}} < t\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right)$$

$$t^{\frac{1}{3}} (x_1)^{\frac{1}{3}} t^{\frac{1}{3}} (x_2)^{\frac{1}{3}} < t\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right)$$

$$t^{\frac{2}{3}} \left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) < t\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right)$$

$$t^{\frac{2}{3}} < t$$

True for any $t > 1$ so this production function has decreasing returns to scale.
$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$ **is monotonic, has diminishing marginal product in both inputs and has decreasing returns to scale.**
For a generic cobb douglass production function $x_1^{\alpha} x_2^{\beta}$

$$(tx_1)^{\alpha} (tx_2)^{\beta} < t\left(x_1^{\alpha} x_2^{\beta}\right)$$

$$t^{\alpha} t^{\beta} (x_1)^{\alpha} (x_2)^{\beta} < t\left(x_1^{\alpha} x_2^{\beta}\right)$$

$$t^{\alpha} t^{\beta} \left(x_1^{\alpha} x_2^{\beta}\right) < t\left(x_1^{\alpha} x_2^{\beta}\right)$$

$$t^{\alpha+\beta} \left(x_1^{\alpha} x_2^{\beta}\right) < t\left(x_1^{\alpha} x_2^{\beta}\right)$$

$$t^{\alpha+\beta} < t$$

This is true for $t > 1$ as long as $\alpha + \beta < 1$.
A cobb douglass production function $x_1^{\alpha} x_2^{\beta}$ is monotonic if $\alpha, \beta > 0$. x_1 has diminishing marginal product if $\alpha < 1$ and x_2 has diminishing marginal product if $\beta < 1$. Decreasing returns to scale when $\alpha + \beta < 1$. Constant returns to scale when $\alpha + \beta = 1$. Increasing returns to scale when $\alpha + \beta > 1$.

$$x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

0.2 Example: Properties of CES

$$x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$$

$$\frac{\partial\left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)}{\partial x_1} = \frac{1}{2\sqrt{x_1}}$$

$$\left((tx_1)^{\frac{1}{2}} + (tx_2)^{\frac{1}{2}}\right) < t\left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)$$

$$t^{\frac{1}{2}}\left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right) < t\left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)$$

$$t^{\frac{1}{2}} < t$$

Monotonic, Diminishing marginal product, Decreasing returns to scale.

1 Profit Maximization / Cost Minimization

1.1 Profit Function

What problem do firms solve?
Output price p. Input prices w_1, w_2.
The firm's revenue is:

$$py = pf(x_1, x_2)$$

The firm's cost is:

$$w_1x_1 + w_2x_2$$

Profit function as a function of the input bundle:

$$\pi(x_1, x_2) = pf(x_1, x_2) - (w_1x_1 + w_2x_2)$$

A firm's goal (objective) is to maximize profit. Pick the bundle inputs that leads to the most profit.

Example. $p = 2$, $f(x_1, x_2) = x_1^{\frac{1}{3}}x_2^{\frac{1}{3}}$ and $w_1 = 1, w_2 = 1$

$$\pi\left(x_1, x_2\right) = 2\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) - x_1 - x_2$$

$$\frac{\partial\left(2\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) - x_1 - x_2\right)}{\partial x_1}$$

$$\frac{\partial\left(2\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) - x_1 - x_2\right)}{\partial x_2}$$

For a function to be at a maximum, all of the partial derivatives must be zero.

$$\frac{\partial\left(2\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) - x_1 - x_2\right)}{\partial x_1} = 0$$

$$\frac{\partial\left(2\left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) - x_1 - x_2\right)}{\partial x_2} = 0$$

$$\frac{2\sqrt[3]{x_2}}{3x_1^{2/3}} - 1 = 0$$

$$\frac{2\sqrt[3]{x_1}}{3x_2^{2/3}} - 1 = 0$$

$$x_1 = \frac{8}{27}, x_2 = \frac{8}{27}$$

$$y = \frac{4}{9}$$

$$\pi^* = 2\frac{4}{9} - \frac{8}{27} - \frac{8}{27} = \frac{8}{27}$$

The most this firm can earn is $\frac{8}{27}$.

1.2 Profit Maximization Requires Cost Minimization

Cost minimization **is** a constrained problem.

What is the bundle of inputs that produces output y and is cheapest among all bundles of inputs that produce y?

Any time a firm is maximizing profit, they have to produce some output y. If x_1 and x_2 aren't the cheapest way of doing this, they can increase profit by continuing to produce y (get the same revenue) but do it in a cheaper way to decrease costs and thus increase profit.

1.3 Two-Step Profit Max

1. Find the cheapest way of producing any level of output y.
2. Choose the optimal output.

1.4 Cost Minimization

Isoquants: Sets of inputs bundles that produce the same output.

$$x_1^{\frac{1}{3}} x_2^{\frac{2}{3}} = \frac{4}{9}$$

Isocost Lines: Sets of input bundles that cost the same.

$$w_1 x_1 + w_2 x_2 = 10$$

The cost minimizing bundle must occur at point where the isocost line through that point is just touching but does pass through the isoquant.

For smooth productions, this occurs where the isoquant and the isocost have the same slope.

1.5 Example- Cobb Douglass

$$f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$$

1.6 Linear/Increasing Returns to Scale

$$f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$$