1 Recap on Cost Minimization

Smooth production:

$$x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

$$TRS = -\frac{w_1}{w_2}$$

Perfect Complements (Non-smooth production). $min\left\{\frac{1}{2}x_1, x_2\right\}$
No-waste condition:

$$\frac{1}{2}x_2 = x_2$$

Linear Production (Perfect Substitutes Production)

$$2x_1 + x_2$$

Look at the cost of producing y using all x_1 or all x_2. Which is cheaper?

2 Example: Perfect Complements

Minimizing the cost of producing output y using the production function. $f(x_1, x_2) = min\left\{\frac{1}{2}x_1, x_2\right\}$ (2 apples and 1 crust). Apples cost 1. $w_1 = 1$ and crusts cost $w_2 = 2$.
The **conditional factor demand** for 1 pie (what bundle of inputs to use to minimize cost of producing 1) is:

$$(2, 1)$$

The conditional factor demand for y pies is:

$$(2y, y)$$

Formally to find the conditional factor demands we use two conditions:
No-waste. $\frac{1}{2}x_1 = x_2$
Production Constraint. $min\left\{\frac{1}{2}x_1, x_2\right\} = y$
Solve these simultaneously:

$$min\left\{\frac{1}{2}x_1, x_2\right\} = y$$

$$min\left\{x_2, x_2\right\} = y$$

$$x_2 = y$$

$$\frac{1}{2}x_1 = y$$

$$x_1 = 2y$$

$$(2y, y)$$

Cost function. (The cost of producing y if you do it in the cheapest way possible.)

$$w_1x_1 + w_2x_2$$

$$w_1(2y) + w_2(y)$$

Since $w_1 = 1$ and $w_2 = 2$.

$$c(y) = 2y + 2y = 4y$$

There is an important relationship between the shape of the function and the shape of the production (returns to scale) at least for homothetic production functions.

For linear returns to scale production, doubling the desired output will double the cost.

For decreasing returns to scale, doubling the desired output will **more than double** the cost.

For **increasing** returns to scale, doubling the desired output will **less than double** the cost.

3 Example: Cobb Douglass

$$f(x_1, x_2) = x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}, w_1 = 1, w_2 = 1$$

$$(tx_1)^{\frac{1}{4}}(tx_2)^{\frac{1}{4}} = t^{\frac{1}{4}}t^{\frac{1}{4}}x_1^{\frac{1}{4}}x_2^{\frac{1}{4}} = t^{\frac{1}{2}}\left(x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}\right)$$

Since $t^{\frac{1}{2}} < t$ for any $t > 1$, this has decreasing returns to scale.
To solve for the conditional factor demands, we need two conditions:
Equal-slope $TRS = -\frac{w_1}{w_2}$

$$-\frac{\frac{\partial\left(x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}\right)}{\partial x_1}}{\frac{\partial\left(x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}\right)}{\partial x_2}} = -\frac{1}{1}$$

$$-\frac{\frac{1}{4}x_1^{-\frac{3}{4}}x_2^{\frac{1}{4}}}{\frac{1}{4}x_2^{-\frac{3}{4}}x_1^{\frac{1}{4}}} = -\frac{1}{1}$$

$$-\frac{x_2}{x_1} = -\frac{1}{1}$$

$$\frac{x_2}{x_1} = 1$$

$$x_2 = x_1$$

Plug this into the production constraint: $x_1^{\frac{1}{4}}x_2^{\frac{1}{4}} = y$

$$x_2^{\frac{1}{4}}x_2^{\frac{1}{4}} = y$$

$$x_2^{\frac{1}{2}} = y$$

$$x_2 = y^2$$

$$x_1 = y^2$$

Confirm this produced y output:

$$\left(y^2\right)^{\frac{1}{4}}\left(y^2\right)^{\frac{1}{4}} = y^{\frac{1}{2}}y^{\frac{1}{2}} = y$$

Cost function:

$$c(y) = y^2 + y^2 = 2y^2$$

Will doubling output more than double cost?

$$c(y) = 2y^2$$

$$c(2y) = 2(2y)^2 = 8y^2$$

Doubling the output increases cost by a factor of 4.

4 Maximizing Profit After Minimizing Cost

Suppose the price of output is $p = 40$. $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$ and $w_1 = 1$, $w_2 = 1$. Writing profit in

$$\pi(x_1, x_2) = 40\left(x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}\right) - (w_1 x_1 + w_2 x_2)$$

$$x_1 \to 100, x_2 \to 100$$

$$f(100, 100) = 100^{\frac{1}{4}} 100^{\frac{1}{4}} = 100^{\frac{1}{2}} = 10$$

How would we do this with the cost function?

$$\pi(y) = py - c(y)$$

$$\pi(y) = 40y - 2y^2$$

At this point we have written down profit purely in terms of y. Solve for where this profit is maximized.

$$\frac{\partial\left(40y - 2y^2\right)}{\partial y} = 0$$

$$40 - 4y = 0$$

$$y = 10$$

$$\pi(10) = 400 - 200 = 200$$

To produce this in the cheapest way: use $x_1 = 100, x_2 = 100$

5 Example: Cobb Douglass Part 2

$$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}, w_1 = 1, w_2 = 1, p = 40$$

$$\pi(x_1, x_2) = 40\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right) - x_1 - x_2$$

$$Rev = 100, Cost = 50, \pi = 50$$

$$Rev = 200, Cost = 100, \pi = 100$$

Let's find the cost minimizing inputs:

$$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}, w_1 = 1, w_2 = 1, p = 40$$

$$\frac{x_2}{x_1} = 1$$

$$x_2 = x_1$$

$$x_2^{\frac{1}{2}} x_2^{\frac{1}{2}} = y$$

$$x_2 = y$$

$$x_1 = y$$

$$c(y) = 2y$$

Let's set up the profit function:

$$\pi(y) = 40y - 2y = 38y$$

6 Pie Baking Example:

$$c(y) = 4y$$

$p = 3$. Profit:

$$\pi(y) = 3y - 4y = -1y$$

Profit maximizing output is $y = 0$

$p = 5$.

$$\pi(y) = 5y - 4y = 1y$$

There is no profit maximizing output.

The reason we "like" decreasing returns to scale is that we don't run into the issue of profit that is never maximized.

7 Short run production.

In the short run, at least one of the inputs is fixed. For instance, let's suppose in our cobb doblass example above we are foced to use $x_2 = 1$. In the short-run, the cost minimizing bundle of inputs is irrelevant. However, since one of the two inputs is fixed, the profit function written in terms of the inputs is simplified and becomes one-dimensional:

$$\pi(x_1, x_2) = 40x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} - x_1 - x_2$$

Becomes:

$$\pi(x_1, 1) = 40x_1^{\frac{1}{4}} (1)^{\frac{1}{4}} - x_1 - 1$$

$$\pi(x_1, 1) = 40x_1^{\frac{1}{4}} - x_1 - 1$$

This is easier to maximize since it is one-dimensional. We look for where it has zero slope in terms of x_1.

$$\frac{\partial\left(40x_1^{\frac{1}{4}} - x_1 - 1\right)}{\partial x_1} = 0$$

$$\frac{10}{x_1^{3/4}} - 1 = 0$$

$$10 = x_1^{3/4}$$

$$x_1 = 10^{\frac{4}{3}} \approx 21.5443$$

Thus, the profit maximizing amount of x_1 is 21.5443. To find the output this produces, plug it back into the (short-run) production function.

$$f\left(10^{\frac{4}{3}}, 1\right) = \left(10^{\frac{4}{3}}\right)^{\frac{1}{4}} (1)^{\frac{1}{4}} = \sqrt[3]{10} \approx 2.15443$$

The profit is:

$$\pi\left(10^{\frac{4}{3}}, 1\right) = 40\left(10^{\frac{4}{3}}\right)^{\frac{1}{4}} (1)^{\frac{1}{4}} - 10^{\frac{4}{3}} - 1 \approx 63.633$$