1 Cournot Model - Two firms.

N firms. q_i is i's quantity. Q is the market quantity $\sum_{i=1}^{N} q_i$. $Q_{-i} = Q - q_i$

$$Q = Q_{-i} + q_i$$

Suppose inverse demand is $p(Q) = 100 - Q$, there are two firms, and the cost function of each firm is $c(q_i) = 10q_i$.

$$\pi_i(q_i, Q_{-i}) = q_i(p(q_i + Q_{-i})) - c(q_i)$$

$$\pi_i(q_i, Q_{-i}) = q_i(100 - (q_i + Q_{-i})) - 10q_i$$

1.1 Monopoly and Perfect Competition Baseline

Perfect competition

$$\pi(q_i) = q_i(p) - 10q_i$$

To maximize this take the derivative with respect to quantity q_i

$$p - 10 = 0$$

$$p = 10$$

$$Q = 90$$

$$\pi_i = 0$$

Monopoly.
Let's drop the i subscript because there is only one firm:

$$\pi(q) = q(100 - q) - 10q$$

To maximize this, we take the derivative with respect to q:

$$\pi(q) = 90q - q^2$$

$$\frac{\partial\left(90q - q^2\right)}{\partial q} = 90 - 2q$$

$$90 = 2q$$

$$q = 45$$

$$p = 55$$

$$\pi(45) = 2025$$

1.2 Equilibrium with 2 firms.

$N = 2$

Here $Q_{-1} = q_2$ and $Q_{-2} = q_1$

$$\pi_1(q_1, q_2) = q_1(100 - (q_1 + q_2)) - 10q_1$$

$$\pi_2(q_2, q_1) = q_2(100 - (q_1 + q_2)) - 10q_2$$

Each firm chooses their own quantity. We need to find the best response function. For any q_2 gives firm 1's optimal q_1. For any q_1 gives firm 2's optimal q_2.

Best response for firm 1 is the profit-maximizing quantity. Find where the derivative is zero:

$$\frac{\partial(q_1(100 - (q_1 + q_2)) - 10q_1)}{\partial q_1} = 0$$

$$-2q_1 - q_2 + 90 = 0$$

Solve for q_1 to get the best response:

$$q_1 = \frac{90 - q_2}{2}$$

If $q_2 = 10$ then firm 1 wants to produce $q_1 = 40$

$$\pi(q_1, 10) = q_1(100 - (q_1 + 10)) - 10q_1$$

10	700
20	1200
30	1500
40	1600
50	1500
60	1200

Firm 2's best response is symmetric:

$$q_2 = \frac{90 - q_1}{2}$$

The best response functions are:

$$q_1 = \frac{90 - q_2}{2}, q_2 = \frac{90 - q_1}{2}$$

Nash Equilibrium. "All players are simultaneously best-responding to eachother".

We need a pair q_1 and a q_2 such that q_1 is a best response to q_2 and q_2 is a best response to q_1.

$$(q_1, 10)$$

$$q_1 = \frac{90 - q_2}{2}, q_2 = \frac{90 - q_1}{2}$$

If q_2 is 10, then q_1 needs to be 45.

$$(45, 10)$$

But this is not an equilibrium. Since $q_2 = 10$ is not a best response to 45 since the best response is $q_2 = \frac{45}{2}$.

To find the equilibrium, we solve the best-response functions simultaneously:

$$q_1 = 30, q_2 = 30$$

Let's check these are best responses to eachother:

$$q_1 = \frac{90 - 30}{2} = 30, q_2 = \frac{90 - 30}{2} = 30$$

This is the only nash equilibrium of the game.

1.3 Different costs (not on the exam)

$$\pi_1 (q_1, q_2) = q_1 (100 - (q_1 + q_2)) - 10q_1$$

$$\frac{\partial (q_1 (100 - (q_1 + q_2)) - 10q_1)}{\partial q_1} = -2q_1 - q_2 + 90$$

$$-2q_1 - q_2 + 90 == 0$$

$$\pi_2 (q_2, q_1) = q_2 (100 - (q_1 + q_2)) - 20q_2$$

$$\frac{\partial (q_2 (100 - (q_1 + q_2)) - 20q_2)}{\partial q_2} = -q_1 - 2q_2 + 80$$

$$-q_1 - 2q_2 + 80 == 0$$

$$q_1 = \frac{100}{3}, q_2 = \frac{70}{3}$$

1.4 A "Hack" for Symmetric Equilibria

$$q_1 (100 - q_1 - q_2) - 10q_1$$

When you impose symmetry on this model, do it on the best response function **not the profit function.**

$$q_1 = \frac{90 - q_2}{2}, q_2 = \frac{90 - q_1}{2}$$

We know there an equilibrium where: $q_1 = q_2 = q$

$$q = \frac{90 - q}{2}, q = \frac{90 - q}{2}$$

Solving this pair of identical equations is the same as solving the one equation:

$$q = \frac{90 - q}{2}$$

$$2q = 90 - q$$

$$q = 30$$

1.5 Equilibrium with N firms.

Now suppose we have N firms.

$$\pi_i (q_i, Q_{-i}) = q_i (100 - (q_i + Q_{-i})) - 10q_i$$

$$= 100q_i - q_i (q_i + Q_{-i}) - 10q_i$$

$$= 100q_i - q_i^2 - q_i Q_{-i} - 10q_i$$

$$\pi_i (q_i, Q_{-i}) = 90q_i - q_i^2 - q_i Q_{-i}$$

Firm i maximizes this with respect to q_i.

$$\frac{\partial \left(90q_i - q_i^2 - q_i Q_{-i}\right)}{\partial q_i} = 0$$

$$90 - 2q_i - Q_{-i} = 0$$

$$q_i = \frac{90 - Q_{-i}}{2}$$

In the equilibrium $q_1 = q_2 = q_3 ... = q_N = q$. $Q = \sum_{i=1}^{N} q = Nq$.

$$Q_{-i} = Nq - q = (N-1)\,q$$

$$q = \frac{90 - (Q_{-i})}{2}$$

$$q = \frac{90 - (N-1)\,q}{2}$$

Solve this to get the nash equilibrium q:

$$q = 45 - \frac{(N-1)\,q}{2}$$

$$q + \frac{(N-1)\,q}{2} = 45$$

$$\frac{2q}{2} + \frac{(N-1)\,q}{2} = 45$$

$$\frac{2q + Nq - q}{2} = 45$$

$$q + Nq = 90$$

$$(N+1)\,q = 90$$

$$q = \frac{90}{N+1}$$

In equilibrium all firms choose $q = \frac{90}{N+1}$.
$N = 1$. $q = \frac{90}{2} = 45$.
$N = 2$. $q = \frac{90}{3} = 30$.
Here is q, Q, p for various N:

N	q	Q	p
1	45.	45.	55.
2	30.	60.	40.
5	15.	75.	25.
10	8.18182	81.8182	18.1818
50	1.76471	88.2353	11.7647
100	0.891089	89.1089	10.8911
100	0.891089	89.1089	10.8911
10000	**0.0089991**	89.991	10.009

The profit of each firm in equilibrium:

$$\pi_i^*(N) = \frac{90}{N+1}\left(100 - \left(N\frac{90}{N+1}\right)\right) - 10\left(\frac{90.0}{N+1}\right)$$

Here is profit of each firm for various N

N	π
1	2025.
2	900.
5	225.
10	66.9421
100	0.79404