# Course Notes: Intermediate Microeconomics

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These notes were created for my "Intermediate Microeconomics" course at Loyola Marymount.

# Contents

# 1 Bundles and Budget

## 1.1 Bundles

A bundle is a vector (an ordered pair or "tuple" of numbers) representing amounts of things. In this class, our models will often only involve two things. Each number in the vector represents an amount of some underlying thing or good. The bundle $(x_1, x_2)$ is a bundle of two goods where $x_1$ represents the amount of good 1 in the bundle and $x_2$ represents the amount of good 2.

> **Definition 1.1: Bundle.**
> A bundle $x$ is an amount of good one $x_1$ and an amount of good two $x_2$ combined into a *vector*. Formally, $x = (x_1, x_2)$

To make this concrete, suppose that we are building a model about the choice of ice cream bowls. If these bowls can only have two flavors: vanilla and chocolate, then the possible bowls can be written as ordered pairs where $x_1$ is the amount of vanilla and $x_2$ is the amount of chocolate.

Here are some possible bundles in this model: $(0, 1)$ zero scoops of vanilla and one scoop of chocolate. $(2, 0)$ two scoops of vanilla and zero scoops of chocolate. $(2, 2)$ two scoops of each flavor.

Since these bundles are ordered pairs or *vectors*, we can plot them. The ice cream examples are plotted in Figure 1.1.

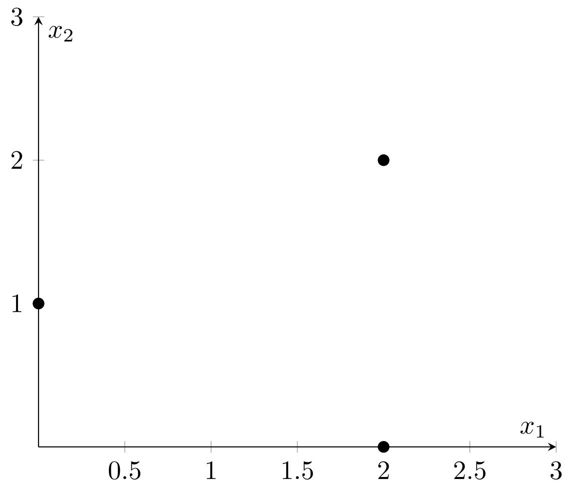



**Figure 1.1: Bundles**

## 1.2 Feasible Set

> **Definition 1.2: The Feasible Set.**
> $X$ is the "feasible" set of bundles.

The feasible set is the universe of bundles that might be relevant in a model. The feasible set defines the scope of a model.

In our ice cream example, the feasible set $X$ might be the set of all bowls that have a non-negative amount of scoops of vanilla and chocolate.

In reality, usually we are limited to choosing an integer amount of scoops of ice cream. However, allowing only integer choices can cause some complexity in analyzing models. For this reason, we often assume that our feasible sets allow bundles with **any** real number of each good. In this case, $x = (1.25, 2.387)$ would be a feasible bundle. Perhaps it is best to think of the quantities of each good as something like ounces of ice cream, instead of an informal unit of measurement like "scoops". So, this bundle would be 1.25 ounces of vanilla and 2.387 ounces of chocolate ice cream much more natural.

## 1.3 Budget Set

**Definition 1.3: Budget Set.**
The **Budget Set** $B$ is the set of all affordable bundles.

While the feasible set $X$ is representative of all possible bundles, the budget set $B$ is the set of bundles *available* for a particular consumer. Not all possible bundles, $X$, will be available to all consumers by default; the budget set, $B$, must be consulted. The budget set must be a subset of the feasible set. In set notation, we write: $B \subseteq X$ which literally says "$B$ is a subset of $X$". The symbol $\subseteq$ allows the two sets to be equal. $B$ does **not** have to be strictly *smaller* than $X$, it just can't be *bigger* than $X$. That is, anything in the budget set actually has to be a feasible bundle.

Budget sets can be anything, really. For instance, the ice cream shop might give you a coupon that says "This coupon entitles you to either 7 ounces of vanilla ice cream *or* 3 ounces vanilla and 4 ounces of chocolate ice cream". This is a weird coupon, but it is perfectly representable with our notation. In this case, your budget set is: $B = \{(7, 0), (3, 4)\}$. Normally, however, our budget sets will be more well-behaved.

## 1.4 Budget Sets from Prices and Income

**Definition 1.4: Prices.**
$p_1$ is the **price of good 1** and $p_2$ is the **price of good 2**.

**Definition 1.5: Income.**
$m$ is **Income**.

Most of the time, we think of "budget" as meaning you have some amount of money you can spend on stuff. That is actually the usual way we will define what $B$ is. We have prices, $p_1$ and $p_2$, and an amount of money to spend, $m$. We usually call $m$ the "income".

To construct the budget set, we will first need to calculate the cost of any bundle: $p_1x_1 + p_2x_2$. From here, the set of bundles that a consumer can have is simply *all* the bundles for which the cost is *less* than $m$. Mathematically: $x_1p_1 + x_2p_2 \leq m$. Thus, we can define if formally this way. The budget set: $B = \{x | x \in X \,\&\, x_1p_1 + x_2p_2 \leq m\}$ . This set theory notation says that "$B$ is the set of bundles $x$ that are both in the feasible set $X$ and such that the price $x_1p_1 + x_2p_2$ of the bundle is less than income $m$.

We will often want to look at the budget graphically. To do this, first we draw the Budget Line. This is the set of bundles that are "just affordable". That is, they cost *exactly* $m$.

> **Definition 1.6: Budget Line.**
> The **Budget Line** is the set of all bundles that cost the full income $m$. This is the set of points on the line:
> $$x_1p_1 + x_2p_2 = m$$

Now we can plot this on an $x_1, x_2$ plane. Let's put $x_2$ of the vertical axis. In this case, it is useful to rewrite the budget line into a form we are more familiar with: $x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$.

This is now clearly an equation for a line with intercept $\frac{m}{p_2}$ and slope $-\frac{p_1}{p_2}$. Before we plot it, let's interpret it a little. Notice that if $x_1 = 0$ we get $x_2 = \frac{m}{p_2}$. This says "If I were only to buy $x_2$, I could afford $\frac{m}{p_2}$ units of $x_2$. Furthermore, for every unit that we increase $x_1$ by, $x_2$ decreases by $-\frac{p_1}{p_2}$. This says "If I am spending all my money and if I want to buy one more unit of $x_1$, I have to give up $-\frac{p_1}{p_2}$ units of $x_2$." This is a very important thing to know about the slope of the budget line. The slope of the budget line represents the trade-off between $x_1$ and $x_2$ at the market prices.

> **Info 1.1: Slope of Budget Line.** The slope of the budget line is $-\frac{p_1}{p_2}$. This slope represents the trade-off between $x_1$ and $x_2$ when a consumer has spent all of their money. It measures how much $x_2$ the consumer must give up to get one more unit of $x_1$.

Two useful bundles on the budget line to know about are the end-points. I recommend always labeling these on a plot of the budget. The two end points are $\left(\frac{m}{p_1}, 0\right)$ and $\left(0, \frac{m}{p_2}\right)$. These are how much $x_1$ the consumer can afford if they buy only $x_1$ and the same is true for $x_2$.

> **Definition 1.7: Budget Set with Prices and Income.**
> When the budget is determined by what bundles can be purchased at prices $p_1, p_2$ with income $m$, the **Budget Set** is the set of all bundles meeting the inequality:
> $$x_1p_1 + x_2p_2 \leq m$$

We are now ready to plot the budget set. The budget set is the budget line *and* all of the bundles "below" the budget line. A budget set is formed this way from prices and income is shown in Figure 1.2.

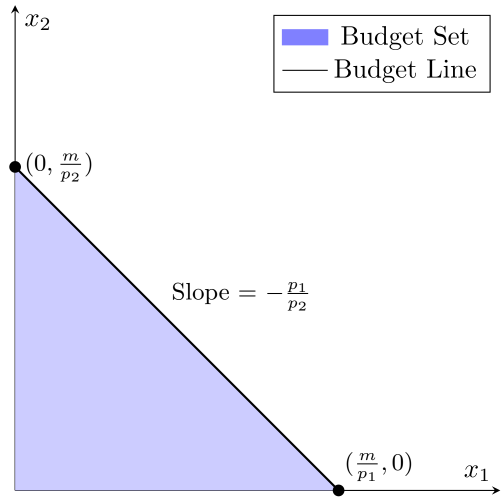



**Figure 1.2: The Budget Set.**

## 1.5 Changing Prices and Income

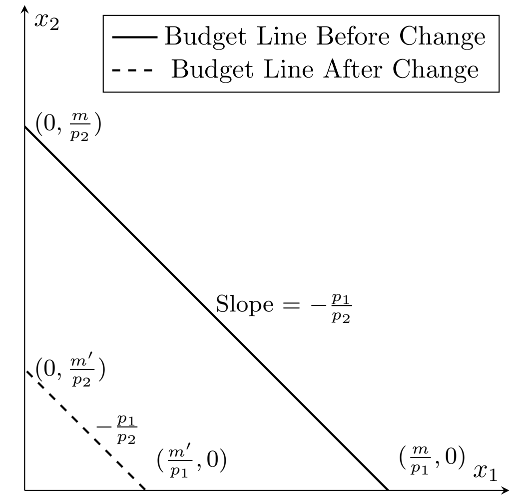
We are often interested in how the budget set changes when we changes one of the budget parameters: $p_1, p_2$ or $m$. Since the budget set is simply all of the bundles on and below the budget line, we will just focus on what happens to the budget line when we change one of these parameters.

It is easy to determine what happens to the budget line by looking at how a change in one of these parameters affects the three key elements of the budget line: the slope, $-\frac{p_1}{p_2}$, the $x_1$ intercept, $\left(\frac{m}{p_1}, 0\right)$, and the $x_2$ intercept $\left(0, \frac{m}{p_2}\right)$.

### 1.5.1 Change in Income

Suppose that income decreases. $m$ changes.

Both endpoints will change. If $m$ increases, $\frac{m}{p_1}$ (the maximum amount I can buy of $x_1$) increases, and $\frac{m}{p_2}$ (the maximum amount I can buy of $x_2$) increases. The slope does not change. If $m$ decreases, the opposite happens. The case of increasing $m$ is shown in the left panel of Figure 1.3.

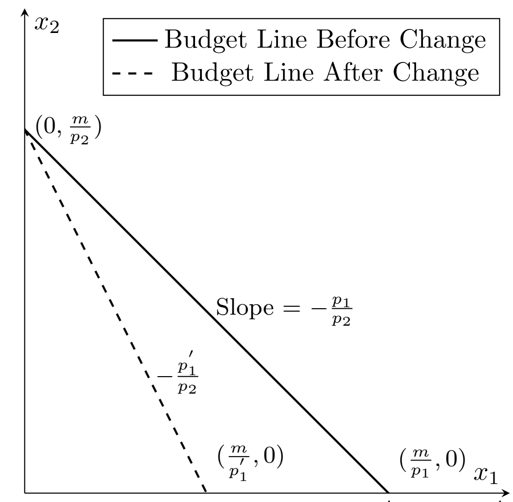



Income changes from $m$ to $m'$ where $m' < m$.

**Figure 1.3: How the Budget Line Changes with a Change in Income**

### 1.5.2 Change in Prices

$p_1$: If $p_1$ *increases*, the slope *decreases* (becomes more negative). If $p_1$ *decreases*, the slope *increases* (becomes more positive). The $x_2$ intercept will stay the same. The case of increasing $p_1$ is shown in the center panel of Figure 1.4.




Price of good 1 changes from $p_1$ to $p_1^{'}$ where $p_1^{'} > p_1$.

**Figure 1.4: How the Budget Line Changes with an Increase in $p_1$**

$p_2$: If $p_2$ *increases*, the slope *increases* as well. If $p_2$ *decreases*, the slope *decreases* as well. The $x_1$ intercept will stay the same. The case of increasing $p_2$ is shown in the right panel of Figure 1.3.

## 1.6 Taxes

Taxes are a familiar way that consumer's budgets are changed. There are two common types of taxes *quantity* and *ad Valorem* taxes.

### 1.6.1 Quantity Tax

In a **quantity tax**, consumers are charged a fixed amount of money per unit of some good they buy. This is the type of tax you pay on gasoline in the United States. At a gas station, the amount of money you pay in tax **per gallon** of gasoline is usually clearly marked. If the price of gas increases, the amount you pay per gallon in tax remains the same.

Suppose a quantity tax of $t$ is added to good 1. The consumer has to pay $tx_1$ in tax. This gives us a new budget of $p_1x_1 + tx_1 + p_2x_2 = m$. This can also be written as $(p_1 + t)\,x_1 + p_2x_2 = m$ which demonstrates very clearly that a quantity tax simply increases the price of a good by $t$. In this case, the introduction of the tax on good one increases the price of good one from $p_1$ to $p_1 + t$.

**Definition 1.8: Quantity Tax on Good 1.**
When there is a **quantity tax** of $t$ on good 1, the budget line is:

$$x_1p_1 + tx_1 + x_2p_2 = m$$

**Definition 1.9: Quantity Tax on Good 2.**
When there is a **quantity tax** of $t$ on good 2, the budget line is:

$$x_1p_1 + x_2p_2 + tx_2 = m$$

### 1.6.2 ad Valorem Tax

**Ad Valorem** taxes are the type of tax you pay in regular sales tax in the United States. An ad Valorem tax is assessed on the *total value* of the goods purchased. For instance, if the tax is 9% then, when you visit a store, you pay 9% of the total cost of goods purchased. Because the tax is assessed based on the cost of the goods if the price of those goods goes up, then amount you pay in tax also increases.

For instance, suppose an ad Valorem of $\tau$ (pronounced "tau") is assessed on good 1. Then if the consumer buys $x_1$ units of good one, the cost is $p_1x_1$. The tax owed is $\tau(p_1x_1)$. This gives us a new budget of $p_1x_1 + \tau\,(p_1x_1) + p_2x_2 = m$. This can also be written as $(1 + \tau)\,p_1x_1 + p_2x_2 = m$. This demonstrates, again, that an ad Valorem tax is just another way of increasing the price of a good.

**Definition 1.10: ad Valorem Tax on Good 1.**
When there is a **ad Valorem tax** of $\tau$ on good 1, the budget line is:

$$(1 + \tau)\,p_1x_1 + p_2x_2 = m$$

*While I think it is important to know about both types of taxes,* ***for mathematical problems we work in this course, we will focus on quantity taxes****. Ultimately, both taxes achieve the same result of increasing the price of a good, but the math tends to be a little easier to work out for quantity taxes.*

### 1.7 Key Topics

- Understand what a bundle is and work with them mathematically as in *Exercise 1.1*.
- Understand the difference between the feasible set and the budget set.
- Understand the difference between the budget set and the budget line.
- Determine whether a bundle is in the budget set or on the budget line as in *Exercise 1.5-1.7*.
- Be able to mathematically express, plot, and work with the budget line and budget set from prices and income as in *Exercises 1.8-1.11*.
- Understand and demonstrate graphically how changes in prices and income affect the budget line and budget set as in *Exercises 1.12,1.13*.
- Understand the **conceptual** difference between a quantity tax and an ad Valorem tax.
- Understand how taxes affect trade-offs along the budget line as in *Exercises 1.4*.
- Mathematically express the budget line and budget set under a **quantity** tax and demonstrate graphically how they affect the budget as in *Exercises 1.14*.

## 2 Binary Relations

### 2.1 Relations

In mathematics, a binary relation is a concept that describes a relationship between things. They allow us to express various kinds of relationships.[1]

> **Definition 2.1: Relation.** A relation on a set $X$ expresses some relationship between the elements of $X$. If $a$ is related to $b$ by the relationship we wish to express, we write $aRb$.[a]
>
> [a]Technically, a relation can be *between* two different sets $A$ and $B$, but in this course we are usually representing relationships among elements of a single set.

While this may seem like a somewhat formal concept, relations are very familiar. Here are some examples of mathematical binary relations.

Suppose $R$ is the *sibling* relation on the set of *all* people in the world. Here, $aRb$ means "person $a$ is a sibling of person $b$". If Laura $l$ and Mike $m$ are siblings, then we can write $lRm$. Similarly $mRl$, as, if Laura is a sibling of Mike, then Mike, by default, is also a sibling of Laura: this relationship is symmetric.

[1]Formally, a **binary relation** $R$ on a set $A$ is a subset of the Cartesian product $A \times A$. That is, $R \subseteq A \times A$. If $(a, b) \in R$, then we say that $a$ is related to $b$ and write $aRb$.

Now, suppose instead $R$ is the *friend* relation on the set of *all* people in the world. Here, $aRb$ means "person $a$ is a friend of person $b$". If Michael $m$ and Sarah $s$ are friends, then $mRs$. Similarly $sRm$, again, due to symmetry; if Michael is a friend of Sarah, then Sarah, by default, is also a friend of Michael.

Once more, suppose instead $R$ is the "at least as tall as" relation on a set of people, where $aRb$ means "person $a$ is at least as tall as person $b$". For example, if John $j$ is *taller* than Alice $a$, then $jRa$. Notice that unlike the previous two examples, we would *not* say $aRj$ since Alice is *not* at least as tall as John, the relationship is asymmetrical (unless they are somehow the exact same height).

In mathematics, things like $\geq, >, \leq, <$, and even $=$ are relations on the set of numbers. We write $4 \geq 3$ since 4 greater than or equal to 3.

As we can see, binary relations can capture a wide range of relationships both in mathematics and in more informal contexts.

## 2.2 Properties of Relations

Notice how the sibling relation and the friend relation have a symmetry to them. If person $a$ is a sibling of person $b$ then person $b$ is also a sibling of $a$. The same is the case with friends (I think). In either case, if $aRb$, then $bRa$. We say that such a relation is **symmetric**. Can you think of some other relations on the set of humans that are symmetric?

On the other hand, there are some relations that are never like this. *Strictly taller than* is a relation that is sort of the opposite of symmetric. If $a$ is strictly taller than $b$ then $b$ cannot be strictly taller than $a$. That would be nonsense! Here, $aRb$ implies $b \not R a$. We call such a relation **asymmetric**.

There are many properties such as symmetric and asymmetric that we should know about. Here is a list of some properties a relation can have.

> **Definition 2.2: Reflexive.** A relation $R$ on a set $A$ is reflexive if every element is related to itself, i.e., **Formally:** $\forall a \in A, aRa$.

On the set of people, the relation *same biological parents as* is a reflexive relation since every person has the same biological parents as themselves. Strictly taller than is *not* a reflexive relation since no one is strictly taller than themselves.

> **Definition 2.3: Complete.** A relation $R$ on a set $A$ is total if every pair of elements is related in at least one direction.
> **Formally:** $\forall a, b \in A$, $aRb$ or $bRa$ (or both).

On the set of all people, *at least as old as* is complete since for every two people, one *has* to be at least as old as the other: a 40-year-old is at least as old as a 30-year-old, for example. *Same biological parents as* is not complete since you can easily find two people, $a$ and $b$, that do not have the same biological parents as each other and thus if $R$ represents this relation, $a \not R b$ and $b \not R a$.

**Definition 2.4: Transitive.** A relation $R$ on a set $A$ is transitive if $a$ is related to $b$ and $b$ is related to $c$, then $a$ is related to $c$.
**Formally:** $\forall a, b, c \in A, aRb \ \& \ bRc \Rightarrow aRc$.

On the set of all people, *same biological parents* is transitive. If $a$ has the same biological parents as $b$ and $b$ has the same biological parents as $c$ then $a$ must have the same biological parents as $c$. *Strictly taller than* is also transitive. On the other hand, *friend of* is not transitive. $a$ could be a friend of $b$ and $b$ a friend of $c$ while $a$ does not even know $c$. Thus, if $R$ represents friendship then $aRb$, $bRc$ but $a\not Rc$. This violates transitivity.

**Definition 2.5: Symmetric.** A relation $R$ on a set $A$ is symmetric if any time $a$ is related to $b$, then $b$ is also related to $a$.
**Formally:** $\forall a, b \in A, aRb \Rightarrow bRa$.

*Same biological parents* is symmetric and so is *same hair color*. If *one* direction is true, then *both* directions are true. On the other hand, *strictly taller than* is *not* symmetric.

**Definition 2.6: Asymmetric.** A relation $R$ on a set $A$ is asymmetric if any time $a$ is related to $b$, then $b$ is *not* related to $a$
**Formally:** $\forall a, b \in A, aRb \Rightarrow b\not Ra$.

*Strictly taller than* is asymmetric since *only* one direction can hold. If $a$ is strictly taller than $b$ then $b$ *cannot* be strictly taller than $a$. On the other hand, *same biological parents as* is not asymmetric.[2]

### 2.3 Key Topics

- Know what a **Relation** is and how it is used to represent relationships formally.
- Know what it means for a relation to be **Reflexive, Complete, Transitive, Symmetric,** and **Asymmetric**.
- Be able to determine whether a familiar relation from everyday life or mathematics is Reflexive, Complete, Transitive, Symmetric, and Asymmetric as in *Exercises 2.1-2.3, 2.6*
- Be able to determine whether a formally described relation is Reflexive, Complete, Transitive, Symmetric, and Asymmetric as in *Exercises 2.4-2.5.*

# 3 Preference Relations

We use relations in economics to represent preferences. A preference relation is a relation on the feasible set $X$ that is intended to describe a consumer's **preferences** over the bundles in $X$. For two bundles, $x$ and $y$, the statement "*$x$ is at least as good as $y$*" is shortened to $x \succsim y$.

The preference relation is a set of statements about pairs of bundles. The statement *"bundle $x$ is preferred to bundle $x'$"* is shortened to $x \succsim x'$.

[2]Technically a relation can be both symmetric and asymmetric, but only if it is what we call the *empty* relation where no element of $X$ is related to any other element of $X$.

Suppose $(x_1, x_2)$ are bundles of $x_1$ scoops of vanilla and $x_2$ scoops of chocolate ice cream. Let's suppose a person likes a scoop of vanilla more than a scoop chocolate. Then the following would be true for them: $(1,0) \succsim (0,1)$. They might also like *any* number of scoops of vanilla to that same number of chocolate. Then the following would also be true of their preferences: $(2,0) \succsim (0,2)$ and $(3,0) \succsim (0,3)$ and $(100,0) \succsim (0,100)$.

## 3.1 Indifference and Strict Preference

Continuing, we will keep using our ice cream example where $(x_1, x_2)$ represent bowls of ice cream $x_1$ scoops of vanilla and $x_2$ scoops of chocolate ice cream.

The following is true for a consumer who does not care about flavor at all, just the total amount of ice cream they get: $(1,0) \succsim (0,1), (0,1) \succsim (1,0)$. Notice that we have both $(1,0) \succsim (0,1)$ and $(0,1) \succsim (1,0)$. That is, a scoop of vanilla is just as good as a scoop of chocolate and a scoop of chocolate is just as good as a scoop of vanilla. When this is the case, we say that the consumer is **indifferent** and write $(1,0) \sim (0,1)$.

**Definition 3.1: Indifference Relation.** $a \sim b$ when $a \succsim b$ and $b \succsim a$.
*a is indifferent to b*

If a consumer is not indifferent between two things, we say that they have a **strict preference**. For example, the same consumer prefers two scoops of vanilla to one scoop but does not prefer one scoop to two. That is $(2,0) \succsim (1,0)$, but $(1,0) \not\succsim (2,0)$. In this case, we say $(2,0)$ is **strictly preferred** to $(1,0)$ and write $(2,0) \succ (1,0)$.

**Definition 3.2: Strict Preference Relation.** $a \succ b$ when $a \succsim b$ and $b \not\succsim a$.
*a is strictly preferred to b*

**Info 3.1: Symmetry of $\sim$.**
$\sim$ is a symmetric relation.

**Info 3.2: Asymmetry of $\succ$.**
$\succ$ is an asymmetric relation.

## 3.2 Rational Preferences

We say that a preference relation is **rational** when it is **complete** and **transitive**.

**Definition 3.3: Rational Preference Relation.** A rational preference relation is a *complete* and *transitive* preference relation $\succsim$.

While we are already familiar with these definitions from the previous chapter, they are so important that I will restate them here in slightly different terms and call them **axioms** which is effectively a formal word for *assumption*. Most of economics is build on the assumption that preferences meet these two conditions.

**Axiom 3.1: Complete.** Preferences are **Complete** when, for every pair of bundles, one of the bundles is strictly better than the other or they are indifferent. That is, for all $x, y \in X$, $x \succsim y$ $y \succsim x$ (or both).

Completeness ensures that consumers have some opinion about comparing every pair of bundles. They can say "I'm indifferent" but not "I don't know".

**Axiom 3.2: Transitive.** Preferences are **Transitive** when, $x \succsim y, y \succsim z$ implies $x \succsim z$

**Info 3.3: What it means to be Rational.**
There is a lot of misunderstanding about the formal meaning of the word rational in economics, even among economists' textbook writers. Rationality has little to do with self-interest, being fully informed, or happiness. Though rationality certainly does not preclude these things.

Rational consumers have preferences. Preferences allow the consumer to rank alternatives (ties are allowed). They can have any ranking they want. Rational consumers choose the highest ranked alternative among the set of alternatives they can afford.

Economists represent these rankings with a utility function that gives higher ranked alternatives a higher score. Representing preferences with a utility function allows economists to use the tools of mathematics to study choices.

## 3.3 Why Complete and Transitive?

You might wonder *why* completeness and transitivity are the two key axioms. Economics is about choice and we assume people draw on their preferences to make these choices. If these two axioms hold, a consumer can always use their preferences to make choices. However, if one of the axioms fails, there may be sets of options which they cannot use their preferences to make a choice from.

To help visualize this, let's look at preferences in terms of a **directed graph**.

**Definition 3.4: Directed Graph.** In mathematics, a **directed graph** is a set of vertices and edges that have a direction.

Here is how we can use a directed graph to visualize a preference relation. Suppose we have the following complete and transitive preference relation on the set $\{a, b, c\}$.

$$a \succsim b, a \succsim c, b \succsim c, a \succsim a, b \succsim b, c \succsim c$$

Below, I have created a directed graph from these preferences where there is an arrow pointing from one letter to another if the first is preferred to the second. For instance, there is an arrow pointing from $a$ to $b$ since $a \succsim b$. We can leave off arrows from each letter to themselves.

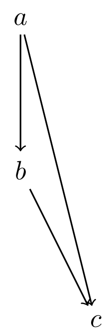



**Figure 3.1: A Complete and Transitive Relation on $\{a, b, c\}$**

> **Info 3.4: Directed Graph from Preference Relation.** To create a directed graph from a preference relation, create a vertex for each object/bundle. If, for two *distinct* bundles $x$ and $y$, $x \succsim y$ then draw a directed edge from $x$ to $y$.

Look how such a complete and transitive relation has a natural ordering. Things higher up in this visualization, like $a$, are better than everything lower. Even when we get more objects and some indifferences, the same kind of shape appears again. Let's look at the directed graph the following complete and transitive relation (it is much easier to visualize with the graph).

$$a \succsim b, b \succsim a, a \succsim c, a \succsim d, b \succsim c, b \succsim d, c \succsim d, d \succsim c, a \succsim e,$$
$$b \succsim e, c \succsim e, d \succsim e, a \succsim a, b \succsim b, c \succsim c, d \succsim d, e \succsim e$$

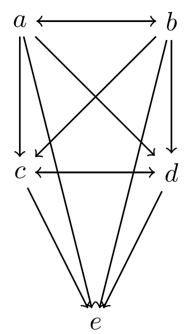



**Figure 3.2: A Complete and Transitive Relation on $\{a, b, c, d, e\}$**

One nice property of such a shape is that, whatever set of objects a consumer might have to choose from, there is some object in that subset that is at least as good as everything else in that set. In the above example, for instance, $a$ and $b$ are at least as good as everything in $\{a, b, c, d, e\}$. $b$ is at least as good as everything in $\{b, d, e\}$. Whatever set we pick, there is at least one object like that. For whatever set the consumer might be asked to choose from, there is at least one **best** object – something they would be happy to choose.

> **Definition 3.5: Best.** $x$ is **best** from some set $B$ (that includes $x$) if $x \succsim y$ for every $y$ in $B$.

We sometimes denote the set of **best** outcomes/options from a set as $C(B)$. This is called the **choice function**. It is the set of best things from a set. Or rather, the things the consumer would be willing to choose. As an example, for the preferences graphed above, $C(a, b, c, d, e) = \{a, b\}$ and $C(b, c, e) = \{b\}$.

What if we make the relation incomplete by removing $a \succsim c$, what is best from the set of $\{a, c\}$? There is nothing as good as everything else, because the consumer has no idea how to compare $a$ and $c$. That is $C(\{a, c\}) = \emptyset$. So, somewhat trivially, when a relation is not complete, there are menus of objects that the consumer cannot choose form. In a less trivial way, this also happens when we make the relation intransitive.

Suppose we have the following complete but **intransitive** preference relation on the set $\{a, b, c\}$.

$$a \succsim b, b \succsim c, c \succsim a, a \succsim a, b \succsim b, c \succsim c$$

.

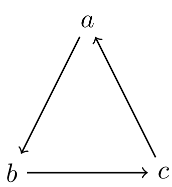



**Figure 3.3: A Complete but Intransitive Relation on $\{a, b, c\}$**

What is best from $\{a, b, c\}$? That is, what is $C(\{a, b, c, \})$? There is nothing at least as good as everything else. $a$ is not at least as good as $c$, $b$ is not at least as good as $a$, $c$ is not at least as good as $b$. What would the consumer choose?!? We have $C(\{a, b, c\}) = \emptyset$. Intransitive preferences create these kinds of cycles (look at the figure again), and when there are cycles, there are sets that the consumer cannot choose from.

## 3.4 Chain Notation

When preferences are complete and transitive there is a more convenient way to express them. Look again at this preference relation:

$$a \succsim b, b \succsim a, a \succsim c, a \succsim d, b \succsim c, b \succsim d, c \succsim d, d \succsim c, a \succsim e,$$
$$b \succsim e, c \succsim e, d \succsim e, a \succsim a, b \succsim b, c \succsim c, d \succsim d, e \succsim e$$

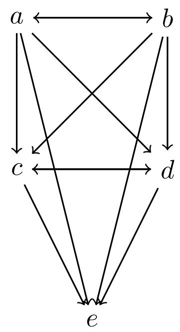



**Figure 3.4: A Complete and Transitive Relation on** $\{a, b, c, d, e\}$

To summarize the preferences in words, $a$ and $b$ are indifferent, but both of these are strictly better than either $c$ or $d$ which are also indifferent, and these in-turn are both strictly better than $e$. We can succinctly summarize this by writing $a \sim b \succ c \sim d \succ e$. This is what I call **chain notation**.

**Definition 3.6: Chain Notation.** Chain notation succinctly summarizes a complete and transitive preference relation. In chain notation, only $\succ$ and $\sim$ are used and the "chain" is arranged such that things that come earlier in the chain are at least as good as anything that comes later in the chain.

Consider these preferences:

$$a \succsim b, a \succsim c, b \succsim c, a \succsim a, b \succsim b, c \succsim c$$

The chain notation for these is $a \succ b \succ c$.

## 3.5 Indifference Curves and Other Sets

For every object, we can use the preference relation to define a few sets. $\succsim (x)$ is the set of objects that is at least as good as $x$. $\succ (x)$ is the set of objects that is strictly better than $x$. $\sim (x)$ is the set of objects indifferent to $x$.

**Definition 3.7: Weakly Preferred Set.** The set of points weakly preferred to $x$ is:

$$\succsim (x) = \{y | y \in X, y \succsim x\}$$

**Definition 3.8: Strictly Preferred Set.** The set of points strictly preferred to $x$ is:

$$\succ (x) = \{y | y \in X, y \succ x\}$$

**Definition 3.9: Indifference Curve.** The set[a] of points indifferent to $x$ is:

$$\sim(x) = \{y | y \in X, y \sim x\}$$

[a] While $\sim (x)$ is also a set of points like $\succsim (x)$ and $\succ (x)$, we usually call $\sim (x)$ and indifference **curve** rather than an indifference set because for most of the examples used in economics the set has no "thickness". It is just... a curve.

Indifference curves are very important in studying preferences. We call such a set of bundles an "indifference curve". We use indifference curves to visualize preferences. Note: There are many indifference curves. We only sketch a few to get an idea of the "shape" of the preferences.

## 3.6 Marginal Rates of Substitution and Slope of the Indifference Curve

**Definition 3.10: Marginal Rate of Substitution.**
The **Marginal Rate of Substitution** measures the slope of an indifference curve at some point. This slope measures, relatively, how much $x_2$ a consumer would give up to get a little more $x_1$.

We will often be a little loose about interpreting the MRS and say that is measures, approximately, how much $x_2$ a consumer will give up to get one more unit of $x_1$. This is not precisely correct since the MRS is a "local" measurement and really measures the relative amount the consumer will give up of $x_2$ for an infinitesimally small increase in $x_1$. However, most importantly, **the MRS measures the willingness to trade-off between good 1 and good 2**.

## 3.7 Examples of Preferences and Their Indifference Curves

### 3.7.1 Perfect Substitutes

**Definition 3.11: Perfect Substitutes.**
Perfect substitutes preferences are preferences where a consumer's willingness to trade off between two goods is always the same. Prefect substitutes preferences have indifference curves that are straight lines.

For example, if a consumer likes apples, but does not care whether an apple is a red apple $x_1$ or green apple $x_2$, they are always willing to give up one $x_2$ to get one $x_1$. Since the slope of the indifference curve encodes this trade-off the indifference curves for this consumer will be lines with slope of $-1$.

For other types of **perfect substitutes** preferences, a steep slope indicates a stronger relative preference for $x_1$. A shallow slope indicates a stronger relative preference for $x_2$.

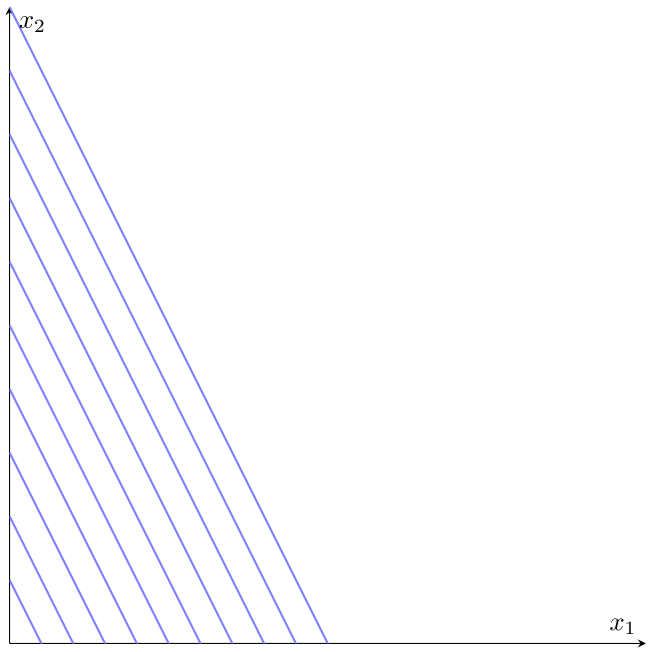



Figure 1: Some indifference curves of a perfect substitutes preference where the consumer will always give up 2 units of $x_2$ to get 1 unit of $x_1$.

#### 3.7.2 Perfect Complements

**Definition 3.12: Perfect Complements.**
Prefect complements represent preferences over goods that must be consumed in fixed proportions. Perfect complements preferences have indifference curves that are L-shaped.

**Prefect complements** preferences represent situations where one good *cannot* substitute for the other. For example, left and right shoes. No matter how many left shoes you have, they cannot replace a right shoe. You must consume them in 1-to-1 combinations.

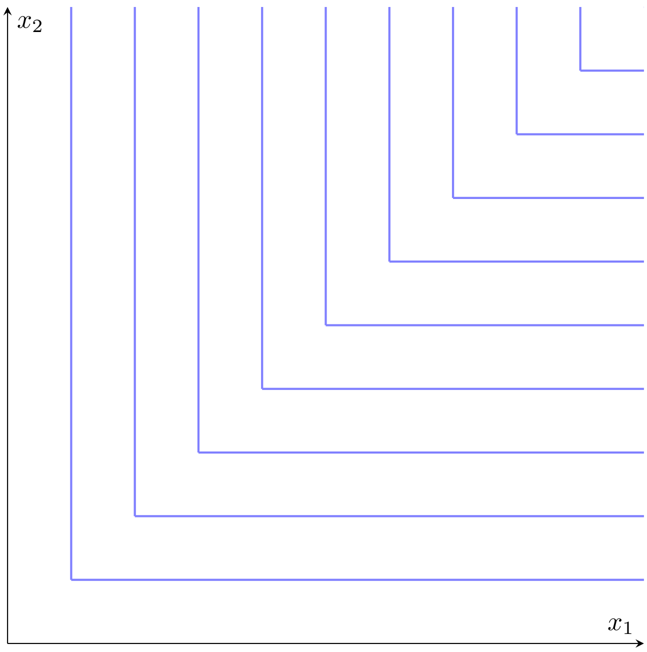



Figure 2: Some indifference curves of a perfect complements preferences where the consumer needs the goods in 1-to-1 combinations.

#### 3.7.3 Bads

**Definition 3.13: Bads.**
If a consumer would prefer to have *less* of a good, then the good is a bad.
If only one of the two goods is bad, the indifference curves slope upward. If both are bad, the indifference curves slope downward, but preference increases as we move toward the point $(0, 0)$.

For example, if a consumer likes ice cream $x_1$, but dislikes broccoli $x_2$ then if you gave them more $x_1$ and took away $x_2$ they would be better off, not indifferent. If you give them more $x_1$ you also have to give them more $x_2$ to make them indifferent. Thus, the indifference curves slope *upwards*.

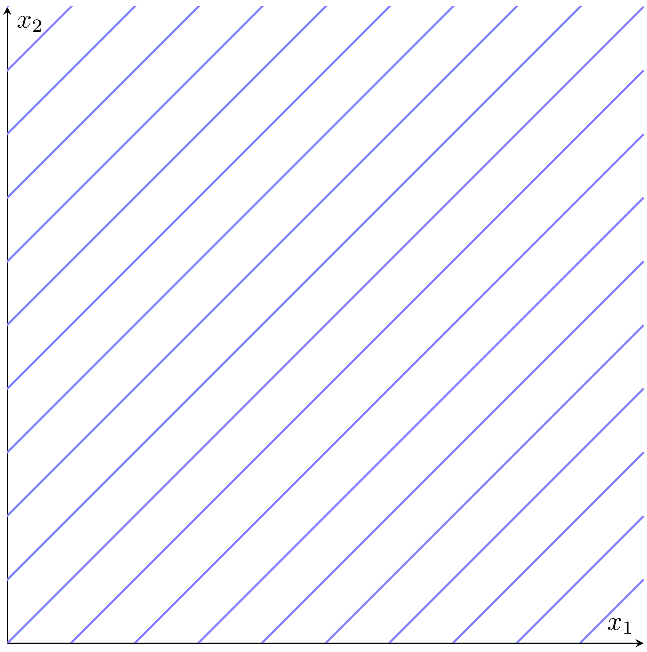



Figure 3: Some indifference curves where $x_2$ is a *bad*.

### 3.8 Indifference Curves Cannot Cross

If preferences are rational, there is one thing we can rule out about the shape of indifference curves: **indifference curves cannot cross**.

> **Info 3.5: Indifference Curves Cannot Cross.**
> If $\succsim$ is rational, two distinct indifference curves cannot cross.

The proof of this is remarkably simple and we will discuss it in class, though *you are not responsible for the proof.*

### 3.9 Key Topics

- Understand how a preference relation is used and defined and how the weak preference relation $\succsim$ can also describe **strict preference** $\succ$ and **indifference** $\sim$.
- Given a weak preference relation, write the strict preference relation as in *Exercise 3.4*.
- Given a weak preference relation, write the indifference relation as in *Exercise 3.5*.
- Understand what properties are needed for a preference relation to be called rational and why those properties are important.
- Given a weak preference relation, determine whether it is complete and transitive as in *Exercise 3.6-3.8*.

- Understand what it means for something to be **best** from a set given a relation and be able to determine what is best from a set given a relation as in *Exercise 3.9*
- Be able to write a preference relation in **chain notation** as in *Exercise 3.3.*
- Understand the definition and use of indifference curves, strictly preferred sets, and weakly preferred sets.
- **Draw** indifference sets, strictly preferred sets, and weakly preferred sets for some described preferences as in *Exercises 3.1-3.2*.
- Understand that the slope of an indifference curve at a bundle is called the **marginal rate of substitution** and that this slope measures a consumer's willingness to trade off between two goods.

# Part I

# Exercises

## 4 Exercises

### 4.1 Exercises for Chapter 1

**Exercise 1.1:** How many units of $x_1$ and $x_2$ are in the bundle $(3, 4)$?

**Exercise 1.2:** List three bundles in the budget set $x_1 + x_2 \leq 5$

**Exercise 1.3:** Sketch the Budget set $B$ that consists of bundles where $x_1 + x_2 \leq 5$.

**Exercise 1.4:** Qualitatively, if a tax is imposed on $x_1$ what does this do to the trade-off between $x_1$ and $x_2$ along th budget line?

For exercises **1.5-1.14**, assume income is $m = 10$ and prices $p_1 = 1$, $p_2 = 2$.

**Exercise 1.5:** Is $(2, 3)$ in the budget set? Is it on the budget line?

**Exercise 1.6:** Is $(1, 5)$ in the budget set? Is it on the budget line?

**Exercise 1.7:** Is $(2, 4)$ in the budget set? Is it on the budget line?

**Exercise 1.8:** Write down the equation of the budget line.

**Exercise 1.9:** How much $x_1$ can the consumer afford if they only buy $x_1$? How about $x_2$?

**Exercise 1.10:** What is the slope of the budget line?

**Exercise 1.11:** Sketch the budget line. Label the slope and endpoints.

**Exercise 1.12:** Plot the budget line again, labeling the slope and endpoints. Demonstrate what happens when $m$ increases to 20. Label the slope and endpoints of the new budget line as well.

**Exercise 1.13:** Plot the budget line again, labeling the slope and endpoints. Demonstrate what happens when $p_1$ increases to 2. Label the slope and endpoints of the new budget line as well.

**Exercise 1.14:** Write down the equation for the budget line if a quantity tax of $t = 2$ is imposed on $x_2$.

## 4.2 Exercises for Chapter 2

*For the relations $R$ below, when a pair is not listed, assume that the relation is not true of that pair.*

**Exercise 2.1:** Is the relation "is a sibling of" on the set of all people:

1. reflexive
2. complete
3. transitive
4. symmetric
5. asymmetric

**Exercise 2.2:** Is the relation "is at least as tall as" on the set of all people:

1. reflexive
2. complete
3. transitive
4. symmetric
5. asymmetric

**Exercise 2.3:** Is the relation "has same birthday as" on the set of all people:

1. reflexive
2. complete
3. transitive
4. symmetric
5. asymmetric

**Exercise 2.4:** For the set $X = \{x, y, z\}$, identify if the following relations are transitive:

1. $R : xRy, yRz, xRz$
2. $R : xRx, yRy, zRz$
3. $R : xRy, yRz, zRx$

**Exercise 2.5:** For the set $X = \{p, q, r\}$, identify if the following relations are complete and transitive. When a relation is not both of these, say which assumption fails and why.

1. $R : pRp, qRq, rRr, pRq, qRr$
2. $R : pRp, qRq, rRr, pRq, qRr, pRr$
3. $R : pRp, qRq, rRr, pRq, qRp, qRr, rRq, pRr, rRp$
4. $R : pRp, qRq, rRr, pRq, qRp, pRr$

**Exercise 2.6:** For the following relations on the set of numbers, determine which of the following properties hold: *reflexive, complete, transitive, symmetric, asymmetric*?

1. $=$
2. $>$
3. $\geq$

**Exercise 2.7: Harder:** Argue that a relation that is complete and symmetric is trivial in the sense that it relates all pairs to each other.

## 4.3 Exercises for Chapter 3

**Exercise 3.1:** Consider the preference relation that describes someone's preferences over left $l$ and right $r$ shoes, where they only care about the number of usable pairs of shoes they consume. Sketch the indifference curves $\sim (1, 1)$ and $\sim (2, 2)$ on graph that has $l$ on the x-axis and $r$ on the y-axis. Label the set $\succ (2, 2)$.

**Exercise 3.2:** Consider the preference relation that describes someone's preferences for red apples $r$ and green apples $g$, where they only care about the total number of apples they have but not the color. Sketch the indifference curves $\sim (1, 1)$ and $\sim (2, 2)$ on graph that has $r$ on the x-axis and $g$ on the y-axis. Label the set $\succ (2, 2)$.

**Exercise 3.3:** Write the following preference relations in **chain notation**.

- $a \succsim b, a \succsim c, b \succsim a, b \succsim c, c \succsim a, c \succsim b, a \succsim a, b \succsim b, c \succsim c$
- $a \succsim b, a \succsim c, b \succsim a, b \succsim c, a \succsim a, b \succsim b, c \succsim c$
- $a \succsim b, a \succsim c, a \succsim d, b \succsim c, b \succsim d, c \succsim b, c \succsim d, a \succsim a, b \succsim b, c \succsim c, d \succsim d$

**Exercise 3.4:** Write the strict preference relation $\succ$ induced by each of the following weak preference relations:

1. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim r, p \succsim r$
2. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim p, q \succsim r, r \succsim q, p \succsim r, r \succsim p$

**Exercise 3.5:** Write the indifference relation $\sim$ induced by each of the following weak preference relations:

1. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim r, p \succsim r$
2. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim p, q \succsim r, r \succsim q, p \succsim r, r \succsim p$

**Exercise 3.6:** Consider the following preference relation on the set $\{a, b, c\}$:

$$a \succsim a, b \succsim b, c \succsim c, a \succsim b, b \succsim c, c \succsim a$$

- Is it complete?
- Is it transitive?

**Exercise 3.7:** Consider the following preference relation on the set $\{a, b, c\}$:

$$a \succsim a, b \succsim b, c \succsim c, a \succsim b, b \succsim a, b \succsim c, c \succsim b, a \succsim c, c \succsim a$$

- Is it complete?
- Is it transitive?

**Exercise 3.8:** Consider the following preference relation on the set $\{a, b, c\}$:

$$a \succsim a, b \succsim b, c \succsim c, b \succsim c, a \succsim c, c \succsim a$$

- Is it complete?
- Is it transitive?

**Exercise 3.9:** For the rational preference relation you plotted in Exercise 3.3:

$$a \succsim b, a \succsim c, a \succsim d, b \succsim c, b \succsim d, c \succsim b, c \succsim d, a \succsim a, b \succsim b, c \succsim c, d \succsim d$$

- What is **best** from set $\{a, b, c, d\}$?
- What is **best** from set $\{b, c, d\}$?
- What is **best** from set $\{c, d\}$?

# 5 Solutions

## 5.1 Solutions for Chapter 1

# Part II

# Appendix

## A Calculus

### A.1 Power Rule

**Info A.1: Power Rule.** The **power rule** states that if $f(x) = x^n$, then the derivative $f'(x) = \frac{\partial f(x)}{\partial x} = nx^{n-1}$.[a]

[a] You may be used to seeing the notation $\frac{df(x)}{dx}$ instead of $\frac{\partial f(x)}{\partial x}$ when taking the derivative of a function with only one variable, but throughout this text I used $\partial$ everywhere to denote a derivative (when there is only one variable) and a partial derivative (when there are multiple variables).

For example, if $f(x) = x^\alpha$ then $f'(x) = \frac{\partial f(x)}{\partial x} = \alpha x^{\alpha-1}$.

### A.2 Derivative of Natural Logarithm

**Info A.2: Derivative of Natural Log.** If $f(x) = \ln(x)$, then the derivative is $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{1}{x}$.

### A.3 Sum Rule

**Info A.3: Sum Rule.** The **sum rule** states that if you have two functions $u(x)$ and $v(x)$, then the derivative of their sum $f(x) = u(x) + v(x)$ is given by:

$$\frac{\partial f(x)}{\partial x} = \frac{\partial u(x)}{\partial x} + \frac{\partial v(x)}{\partial x}$$

For example, if $f(x) = x^2 + \ln(x)$, then the derivative is given by:

$$\frac{\partial f(x)}{\partial x} = 2x + \frac{1}{x}$$

## A.4 Product Rule

**Info A.4: Product Rule.** The **product rule** is a formula used to find the derivative of the product of two functions. If you have two functions $u(x)$ and $v(x)$, then the derivative of their product $y = u(x)v(x)$ is given by:

$$\frac{\partial y}{\partial x} = u(x) \cdot \frac{\partial v}{\partial x} + v(x) \cdot \frac{\partial u}{\partial x}$$

For example, if $f(x) = x^2 \cdot \ln(x)$, then the derivative is given by:

$$\frac{\partial f(x)}{\partial x} = x^2 \cdot \frac{1}{x} + \ln(x) \cdot 2x = x + 2x\ln(x)$$

## A.5 Chain Rule

**Info A.5: Chain Rule.** The **a** states that if you have a composite function $y = g(f(x))$, then the derivative $\frac{\partial y}{\partial x}$ is given by $\frac{\partial y}{\partial f} \cdot \frac{\partial f}{\partial x}$.

For example, if $f(x) = ln(x^2)$ then $f'(x) = \frac{\partial f(x)}{\partial x} = \left(\frac{1}{x^2}\right) 2x = \frac{2}{x}$.

## A.6 Partial Derivatives

When dealing with functions of more than one variable, partial derivatives provide a way to explore how changes in one input variable impact the function, keeping others fixed.

**Info A.6: Partial Derivatives.** A **partial derivative** of a function of multiple variables is its derivative with respect to one of those variables, with the other variables held constant (imagine it is some fixed number). If you have a function $f(x, y)$, then the partial derivative of $f$ with respect to $x$ is denoted as $\frac{\partial z}{\partial x}$, and with respect to $y$ is $\frac{\partial z}{\partial y}$.

For example, if $f(x, y) = x^2y + y^3$, the partial derivative of $f$ with respect to $x$ is $\frac{\partial f}{\partial x} = 2xy$, and with respect to $y$ is $\frac{\partial f}{\partial y} = x^2 + 3y^2$.

## A.7 Second Derivative

The **second derivative** provides information about the curvature of a function. It measures how the first derivative (the slope) changes as a variable changes.

**Info A.7: Second Derivative.** For a function $f(x)$ with a first derivative $f'(x)$, then the second derivative $f''(x) = \frac{\partial^2 f(x)}{\partial x^2}$ is simply the of the $f'(x)$.

For example, if $f(x) = x^3$, then the first derivative is $f'(x) = 3x^2$ and the second derivative is $f''(x) = 6x$.

## A.8 Exercises for Appendix Chapter A

**Exercise A.1:** Find the derivative of $f(x) = x^5$.

**Exercise A.2:** Find the derivative of $f(x) = 2x^7 + 3x^4$ using the power rule.

**Exercise A.3:** Find the derivative of $f(x) = \frac{10}{x^4}$.

**Exercise A.4:** Find the derivative of $f(x) = \ln(x^3 + 1)$.

**Exercise A.5:** Find the derivative of $f(x) = \sqrt{\ln(x)}$.

**Exercise A.6:** Find the partial derivatives of $f(x, y) = x^3+y^3+3xy$ with respect to both $x$ and $y$.

**Exercise A.7:** Find the partial derivatives of $f(x, y) = xy$ with respect to both $x$ and $y$.

**Exercise A.8:** Find the partial derivatives of $f(x, y) = x^2y^2$ with respect to both $x$ and $y$.

**Exercise A.9:** Find the partial derivatives of $f(x, y) = x^\alpha y^\beta$ with respect to both $x$ and $y$.

**Exercise A.10:** Find the partial derivatives of $f(x, y) = x^2y^3+x^3y$ with respect to both $x$ and $y$.

**Exercise A.11:** Find the derivative of $f(x) = \frac{x^5}{x^2+1}$.

**Exercise A.12:** Find the partial derivatives of $f(x, y) = x\ln(y) + y\ln(x)$ with respect to both $x$ and $y$.

**Exercise A.13:** Find the second derivative of $f(x) = x^6 + 5x^2$.

**Exercise A.14:** Find the second derivative of $f(x) = ln(x)$.

## References