Course Notes: Intermediate Microeconomics

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1 Bundles and Budget

1.1 Bundles

A bundle is a vector (an ordered pair or "tuple" of numbers) representing amounts of things. In this class, our models will often only involve two things. Each number in the vector represents an amount of some underlying thing or good. The bundle (x_1, x_2) is a bundle of two goods where x_1 represents the amount of good 1 in the bundle and x_2 represents the amount of good 2.

Definition 1.1: Bundle.
A bundle x is an amount of good one x_1 and an amount of good two x_2 combined into a *vector*. Formally, $x = (x_1, x_2)$

To make this concrete, suppose that we are building a model about the choice of ice cream bowls. If these bowls can only have two flavors: vanilla and chocolate, then the possible bowls can be written as ordered pairs where x_1 is the amount of vanilla and x_2 is the amount of chocolate.

Here are some possible bundles in this model: $(0, 1)$ zero scoops of vanilla and one scoop of chocolate. $(2, 0)$ two scoops of vanilla and zero scoops of chocolate. $(2, 2)$ two scoops of each flavor.

Since these bundles are ordered pairs or *vectors*, we can plot them. The ice cream examples are plotted in Figure 1.1.

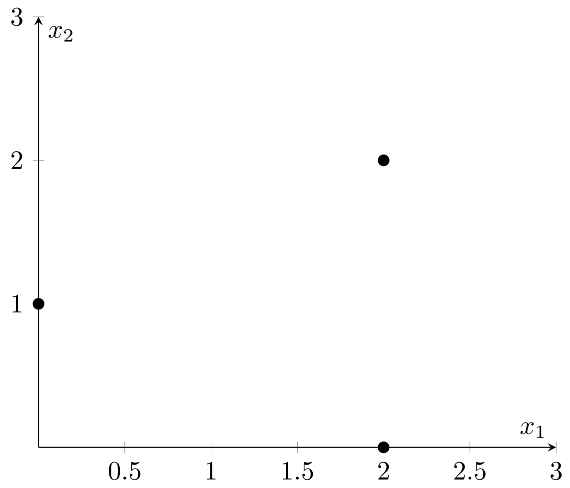


Figure 1.1: Bundles

1.2 Feasible Set

Definition 1.2: The Feasible Set.
X is the "feasible" set of bundles.

The feasible set is the universe of bundles that might be relevant in a model. The feasible set defines the scope of a model.

In our ice cream example, the feasible set X might be the set of all bowls that have a non-negative amount of scoops of vanilla and chocolate.

In reality, usually we are limited to choosing an integer amount of scoops of ice cream. However, allowing only integer choices can cause some complexity in analyzing models. For this reason, we often assume that our feasible sets allow bundles with **any** real number of each good. In this case, $x = (1.25, 2.387)$ would be a feasible bundle. Perhaps it is best to think of the quantities of each good as something like ounces of ice cream, instead of an informal unit of measurement like "scoops". So, this bundle would be 1.25 ounces of vanilla and 2.387 ounces of chocolate ice cream much more natural.

1.3 Budget Set

Definition 1.3: Budget Set.
The **Budget Set** B is the set of all affordable bundles.

While the feasible set X is representative of all possible bundles, the budget set B is the set of bundles *available* for a particular consumer. Not all possible bundles, X, will be available to all consumers by default; the budget set, B, must be consulted. The budget set must be a subset of the feasible set. In set notation, we write: $B \subseteq X$ which literally says "B is a subset of X". The symbol $\subseteq$ allows the two sets to be equal. B does **not** have to be strictly *smaller* than X, it just can't be *bigger* than X. That is, anything in the budget set actually has to be a feasible bundle.

Budget sets can be anything, really. For instance, the ice cream shop might give you a coupon that says "This coupon entitles you to either 7 ounces of vanilla ice cream *or* 3 ounces vanilla and 4 ounces of chocolate ice cream". This is a weird coupon, but it is perfectly representable with our notation. In this case, your budget set is: $B = \{(7, 0), (3, 4)\}$. Normally, however, our budget sets will be more well-behaved.

1.4 Budget Sets from Prices and Income

Definition 1.4: Prices.
p_1 is the **price of good 1** and p_2 is the **price of good 2**.

Definition 1.5: Income.
m is **Income**.

Most of the time, we think of "budget" as meaning you have some amount of money you can spend on stuff. That is actually the usual way we will define what B is. We have prices, p_1 and p_2, and an amount of money to spend, m. We usually call m the "income".

To construct the budget set, we will first need to calculate the cost of any bundle: $p_1x_1 + p_2x_2$. From here, the set of bundles that a consumer can have is simply *all* the bundles for which the cost is *less* than m. Mathematically: $x_1p_1 + x_2p_2 \leq m$. Thus, we can define if formally this way. The budget set: $B = \{x | x \in X \,\&\, x_1p_1 + x_2p_2 \leq m\}$. This set theory notation says that "B is the set of bundles x that are both in the feasible set X and such that the price $x_1p_1 + x_2p_2$ of the bundle is less than income m.

We will often want to look at the budget graphically. To do this, first we draw the Budget Line. This is the set of bundles that are "just affordable". That is, they cost *exactly* m.

Definition 1.6: Budget Line.
The **Budget Line** is the set of all bundles that cost the full income m. This is the set of points on the line:

$$x_1p_1 + x_2p_2 = m$$

Now we can plot this on an x_1, x_2 plane. Let's put x_2 of the vertical axis. In this case, it is useful to rewrite the budget line into a form we are more familiar with: $x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$.

This is now clearly an equation for a line with intercept $\frac{m}{p_2}$ and slope $-\frac{p_1}{p_2}$. Before we plot it, let's interpret it a little. Notice that if $x_1 = 0$ we get $x_2 = \frac{m}{p_2}$. This says "If I were only to buy x_2, I could afford $\frac{m}{p_2}$ units of x_2. Furthermore, for every unit that we increase x_1 by, x_2 decreases by $-\frac{p_1}{p_2}$. This says "If I am spending all my money and if I want to buy one more unit of x_1, I have to give up $-\frac{p_1}{p_2}$ units of x_2." This is a very important thing to know about the slope of the budget line. The slope of the budget line represents the trade-off between x_1 and x_2 at the market prices.

Info 1.1: Slope of Budget Line. The slope of the budget line is $-\frac{p_1}{p_2}$. This slope represents the trade-off between x_1 and x_2 when a consumer has spent all of their money. It measures how much x_2 the consumer must give up to get one more unit of x_1.

Two useful bundles on the budget line to know about are the end-points. I recommend always labeling these on a plot of the budget. The two end points are $\left(\frac{m}{p_1}, 0\right)$ and $\left(0, \frac{m}{p_2}\right)$. These are how much x_1 the consumer can afford if they buy only x_1 and the same is true for x_2.

Definition 1.7: Budget Set with Prices and Income.
When the budget is determined by what bundles can be purchased at prices p_1, p_2 with income m, the **Budget Set** is the set of all bundles meeting the inequality:

$$x_1p_1 + x_2p_2 \leq m$$

We are now ready to plot the budget set. The budget set is the budget line *and* all of the bundles "below" the budget line. A budget set is formed this way from prices and income is shown in Figure 1.2.

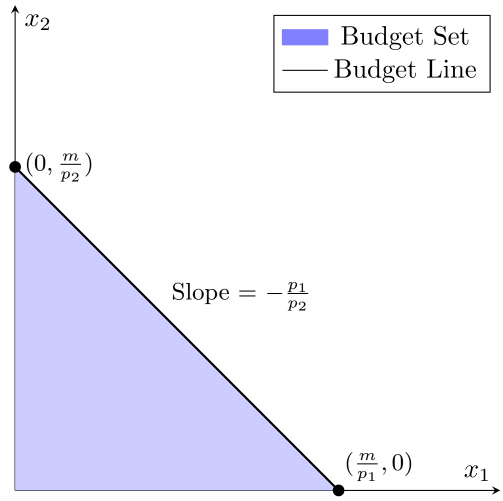


Figure 1.2: The Budget Set.

1.5 Changing Prices and Income

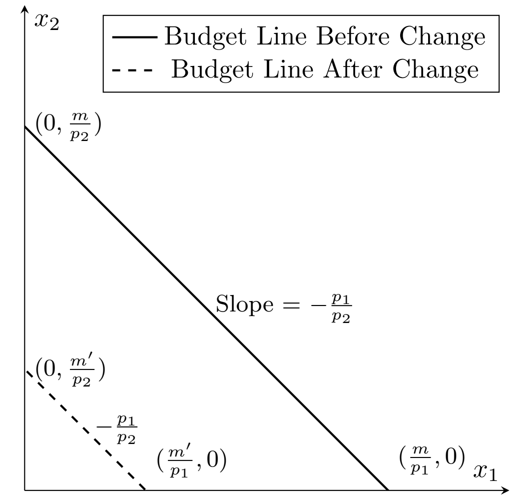
We are often interested in how the budget set changes when we changes one of the budget parameters: p_1, p_2 or m. Since the budget set is simply all of the bundles on and below the budget line, we will just focus on what happens to the budget line when we change one of these parameters.

It is easy to determine what happens to the budget line by looking at how a change in one of these parameters affects the three key elements of the budget line: the slope, $-\frac{p_1}{p_2}$, the x_1 intercept, $\left(\frac{m}{p_1}, 0\right)$, and the x_2 intercept $\left(0, \frac{m}{p_2}\right)$.

1.5.1 Change in Income

Suppose that income decreases. m changes.

Both endpoints will change. If m increases, $\frac{m}{p_1}$ (the maximum amount I can buy of x_1) increases, and $\frac{m}{p_2}$ (the maximum amount I can buy of x_2) increases. The slope does not change. If m decreases, the opposite happens. The case of increasing m is shown in the left panel of Figure 1.3.

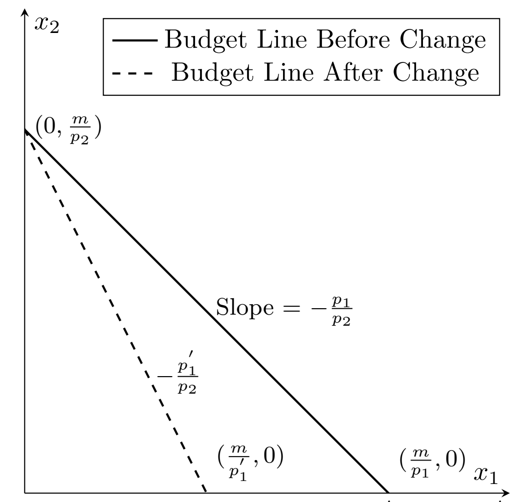


Income changes from m to m' where $m' < m$.

Figure 1.3: How the Budget Line Changes with a Change in Income

1.5.2 Change in Prices

p_1: If p_1 *increases*, the slope *decreases* (becomes more negative). If p_1 *decreases*, the slope *increases* (becomes more positive). The x_2 intercept will stay the same. The case of increasing p_1 is shown in the center panel of Figure 1.4.



Price of good 1 changes from p_1 to $p_1^{'}$ where $p_1^{'} > p_1$.

Figure 1.4: How the Budget Line Changes with an Increase in p_1

p_2: If p_2 *increases*, the slope *increases* as well. If p_2 *decreases*, the slope *decreases* as well. The x_1 intercept will stay the same. The case of increasing p_2 is shown in the right panel of Figure 1.3.

1.6 Taxes

Taxes are a familiar way that consumer's budgets are changed. There are two common types of taxes *quantity* and *ad Valorem* taxes.

1.6.1 Quantity Tax

In a **quantity tax**, consumers are charged a fixed amount of money per unit of some good they buy. This is the type of tax you pay on gasoline in the United States. At a gas station, the amount of money you pay in tax **per gallon** of gasoline is usually clearly marked. If the price of gas increases, the amount you pay per gallon in tax remains the same.

Suppose a quantity tax of t is added to good 1. The consumer has to pay tx_1 in tax. This gives us a new budget of $p_1x_1 + tx_1 + p_2x_2 = m$. This can also be written as $(p_1 + t)\, x_1 + p_2x_2 = m$ which demonstrates very clearly that a quantity tax simply increases the price of a good by t. In this case, the introduction of the tax on good one increases the price of good one from p_1 to $p_1 + t$.

Definition 1.8: Quantity Tax on Good 1.
When there is a **quantity tax** of t on good 1, the budget line is:

$$x_1p_1 + tx_1 + x_2p_2 = m$$

Definition 1.9: Quantity Tax on Good 2.
When there is a **quantity tax** of t on good 2, the budget line is:

$$x_1p_1 + x_2p_2 + tx_2 = m$$

1.6.2 ad Valorem Tax

Ad Valorem taxes are the type of tax you pay in regular sales tax in the United States. An ad Valorem tax is assessed on the *total value* of the goods purchased. For instance, if the tax is 9% then, when you visit a store, you pay 9% of the total cost of goods purchased. Because the tax is assessed based on the cost of the goods if the price of those goods goes up, then amount you pay in tax also increases.

For instance, suppose an ad Valorem of τ (pronounced "tau") is assessed on good 1. Then if the consumer buys x_1 units of good one, the cost is p_1x_1. The tax owed is $\tau(p_1x_1)$. This gives us a new budget of $p_1x_1 + \tau\,(p_1x_1) + p_2x_2 = m$. This can also be written as $(1 + \tau)\, p_1x_1 + p_2x_2 = m$. This demonstrates, again, that an ad Valorem tax is just another way of increasing the price of a good.

Definition 1.10: ad Valorem Tax on Good 1.
When there is a **ad Valorem tax** of τ on good 1, the budget line is:

$$(1 + \tau)\, p_1x_1 + p_2x_2 = m$$

*While I think it is important to know about both types of taxes, **for mathematical problems we work in this course, we will focus on quantity taxes**. Ultimately, both taxes achieve the same result of increasing the price of a good, but the math tends to be a little easier to work out for quantity taxes.*

1.7 Key Topics

- Understand what a bundle is and work with them mathematically as in *Exercise 1.1*.
- Understand the difference between the feasible set and the budget set.
- Understand the difference between the budget set and the budget line.
- Determine whether a bundle is in the budget set or on the budget line as in *Exercise 1.5-1.7*.
- Be able to mathematically express, plot, and work with the budget line and budget set from prices and income as in *Exercises 1.8-1.11*.
- Understand and demonstrate graphically how changes in prices and income affect the budget line and budget set as in *Exercises 1.12,1.13*.
- Understand the **conceptual** difference between a quantity tax and an ad Valorem tax.
- Understand how taxes affect trade-offs along the budget line as in *Exercises 1.4*.
- Mathematically express the budget line and budget set under a **quantity** tax and demonstrate graphically how they affect the budget as in *Exercises 1.14*.

2 Binary Relations

2.1 Relations

In mathematics, a binary relation is a concept that describes a relationship between things. They allow us to express various kinds of relationships.[1]

> **Definition 2.1: Relation.** A relation on a set X expresses some relationship between the elements of X. If a is related to b by the relationship we wish to express, we write aRb.[a]
>
> [a]Technically, a relation can be *between* two different sets A and B, but in this course we are usually representing relationships among elements of a single set.

While this may seem like a somewhat formal concept, relations are very familiar. Here are some examples of mathematical binary relations.

Suppose R is the *sibling* relation on the set of *all* people in the world. Here, aRb means "person a is a sibling of person b". If Laura l and Mike m are siblings, then we can write lRm. Similarly mRl, as, if Laura is a sibling of Mike, then Mike, by default, is also a sibling of Laura: this relationship is symmetric.

Now, suppose instead R is the *friend* relation on the set of *all* people in the world. Here, aRb means "person a is a friend of person b". If Michael m and Sarah s are friends, then mRs. Similarly sRm, again, due to symmetry; if Michael is a friend of Sarah, then Sarah, by default, is also a friend of Michael.

Once more, suppose instead R is the "at least as tall as" relation on a set of people, where aRb means "person a is at least as tall as person b". For example, if John j is *taller* than Alice a, then jRa. Notice that unlike the previous two examples, we would *not* say aRj since Alice is *not* at least as tall as John, the relationship is asymmetrical (unless they are somehow the exact same height).

In mathematics, things like $\geq, >, \leq, <$, and even $=$ are relations on the set of numbers. We write $4 \geq 3$ since 4 greater than or equal to 3.

As we can see, binary relations can capture a wide range of relationships both in mathematics and in more informal contexts.

2.2 Properties of Relations

Notice how the sibling relation and the friend relation have a symmetry to them. If person a is a sibling of person b then person b is also a sibling of a. The same is the case with friends (I think). In either case, if aRb, then bRa. We say that such a relation is **symmetric**. Can you think of some other relations on the set of humans that are symmetric?

On the other hand, there are some relations that are never like this. *Strictly taller than* is a relation that is sort of the opposite of symmetric. If a is strictly taller than b then b cannot be

[1]Formally, a **binary relation** R on a set A is a subset of the Cartesian product $A \times A$. That is, $R \subseteq A \times A$. If $(a, b) \in R$, then we say that a is related to b and write aRb.

strictly taller than a. That would be nonsense! Here, aRb implies $b\not R a$. We call such a relation **asymmetric**.

There are many properties such as symmetric and asymmetric that we should know about. Here is a list of some properties that a relation can have.

> **Definition 2.2: Reflexive.** A relation R on a set A is reflexive if every element is related to itself, i.e., **Formally:** for all $a \in A, aRa$.

On the set of people, the relation *same biological parents as* is a reflexive relation since every person has the same biological parents as themselves. Strictly taller than is *not* a reflexive relation since no one is strictly taller than themselves.

> **Definition 2.3: Complete.** A relation R on a set A is total if every pair of elements is related in at least one direction.
> **Formally:** for all $a, b \in A$, aRb or bRa (or both).

On the set of all people, *at least as old as* is complete since for every two people, one *has* to be at least as old as the other: a 40-year-old is at least as old as a 30-year-old, for example. *Same biological parents as* is not complete since you can easily find two people, a and b, that do not have the same biological parents as each other and thus if R represents this relation, $a\not R b$ and $b\not R a$.

> **Definition 2.4: Transitive.** A relation R on a set A is transitive if a is related to b and b is related to c, then a is related to c.
> **Formally:** for all $a, b, c \in A, aRb$ & $bRc \Rightarrow aRc$.

On the set of all people, *same biological parents* is transitive. If a has the same biological parents as b and b has the same biological parents as c then a must have the same biological parents as c. *Strictly taller than* is also transitive. On the other hand, *friend of* is not transitive. a could be a friend of b and b a friend of c while a does not even know c. Thus, if R represents friendship then aRb, bRc but $a\not R c$. This violates transitivity.

> **Definition 2.5: Symmetric.** A relation R on a set A is symmetric if any time a is related to b, then b is also related to a.
> **Formally:** for all $a, b \in A, aRb \Rightarrow bRa$.

Same biological parents is symmetric and so is *same hair color*. If *one* direction is true, then *both* directions are true. On the other hand, *strictly taller than* is *not* symmetric.

> **Definition 2.6: Asymmetric.** A relation R on a set A is asymmetric if any time a is related to b, then b is *not* related to a
> **Formally:** for all $a, b \in A, aRb \Rightarrow b\not R a$.

Strictly taller than is asymmetric since *only* one direction can hold. If a is strictly taller than b then b *cannot* be strictly taller than a. On the other hand, *same biological parents as* is not asymmetric.[2]

[2] Technically a relation can be both symmetric and asymmetric, but only if it is what we call the *empty* relation

2.3 Key Topics

- Know what a **Relation** is and how it is used to represent relationships formally.
- Know what it means for a relation to be **Reflexive, Complete, Transitive, Symmetric,** and **Asymmetric**.
- Be able to determine whether a familiar relation from everyday life or mathematics is Reflexive, Complete, Transitive, Symmetric, and Asymmetric as in *Exercises 2.1-2.3, 2.6*
- Be able to determine whether a formally described relation is Reflexive, Complete, Transitive, Symmetric, and Asymmetric as in *Exercises 2.4-2.5*.

where no element of X is related to any other element of X.

3 Preference Relations

We use relations in economics to represent preferences. A preference relation is a relation on the feasible set X that is intended to describe a consumer's **preferences** over the bundles in X. For two bundles, x and y, the statement "*x is at least as good as y*" is shortened to $x \succsim y$.

The preference relation is a set of statements about pairs of bundles. The statement *"bundle x is preferred to bundle x'"* is shortened to $x \succsim x'$.

Suppose (x_1, x_2) are bundles of x_1 scoops of vanilla and x_2 scoops of chocolate ice cream. Let's suppose a person likes a scoop of vanilla more than a scoop chocolate. Then the following would be true for them: $(1, 0) \succsim (0, 1)$. They might also like *any* number of scoops of vanilla to that same number of chocolate. Then the following would also be true of their preferences: $(2, 0) \succsim (0, 2)$ and $(3, 0) \succsim (0, 3)$ and $(100, 0) \succsim (0, 100)$.

3.1 Indifference and Strict Preference

Continuing, we will keep using our ice cream example where (x_1, x_2) represent bowls of ice cream x_1 scoops of vanilla and x_2 scoops of chocolate ice cream.

The following is true for a consumer who does not care about flavor at all, just the total amount of ice cream they get: $(1, 0) \succsim (0, 1), (0, 1) \succsim (1, 0)$. Notice that we have both $(1, 0) \succsim (0, 1)$ and $(0, 1) \succsim (1, 0)$. That is, a scoop of vanilla is just as good as a scoop of chocolate and a scoop of chocolate is just as good as a scoop of vanilla. When this is the case, we say that the consumer is **indifferent** and write $(1, 0) \sim (0, 1)$.

Definition 3.1: Indifference Relation. $a \sim b$ when $a \succsim b$ and $b \succsim a$.
a is indifferent to b

If a consumer is not indifferent between two things, we say that they have a **strict preference**. For example, the same consumer prefers two scoops of vanilla to one scoop but does not prefer one scoop to two. That is $(2, 0) \succsim (1, 0)$, but $(1, 0) \not\succsim (2, 0)$. In this case, we say $(2, 0)$ is **strictly preferred** to $(1, 0)$ and write $(2, 0) \succ (1, 0)$.

Definition 3.2: Strict Preference Relation. $a \succ b$ when $a \succsim b$ and $b \not\succsim a$.
a is strictly preferred to b

Info 3.1: Symmetry of $\sim$.
$\sim$ is a symmetric relation.

Info 3.2: Asymmetry of $\succ$.
$\succ$ is an asymmetric relation.

3.2 Rational Preferences

We say that a preference relation is **rational** when it is **complete** and **transitive**.

Definition 3.3: Rational Preference Relation. A rational preference relation is a *complete* and *transitive* preference relation $\succsim$.

While we are already familiar with these definitions from the previous chapter, they are so important that I will restate them here in slightly different terms and call them **axioms** which is effectively a formal word for *assumption.* Most of economics is build on the assumption that preferences meet these two conditions.

Axiom 3.1: Complete. Preferences are **Complete** when, for every pair of bundles, one of the bundles is strictly better than the other or they are indifferent. That is, for all $x, y \in X$, $x \succsim y$ $y \succsim x$ (or both).

Completeness ensures that consumers have some opinion about comparing every pair of bundles. They can say "I'm indifferent" but not "I don't know".

Axiom 3.2: Transitive. Preferences are **Transitive** when, $x \succsim y, y \succsim z$ implies $x \succsim z$

Info 3.3: What it means to be Rational.
There is a lot of misunderstanding about the formal meaning of the word rational in economics, even among economists' textbook writers. Rationality has little to do with self-interest, being fully informed, or happiness. Though rationality certainly does not preclude these things.

Rational consumers have preferences. Preferences allow the consumer to rank alternatives (ties are allowed). They can have any ranking they want. Rational consumers choose the highest ranked alternative among the set of alternatives they can afford.

Economists represent these rankings with a utility function that gives higher ranked alternatives a higher score. Representing preferences with a utility function allows economists to use the tools of mathematics to study choices.

3.3 Why Complete and Transitive?

You might wonder *why* completeness and transitivity are the two key axioms. Economics is about choice and we assume people draw on their preferences to make these choices. If these two axioms hold, a consumer can always use their preferences to make choices. However, if one of the axioms fails, there may be sets of options which they cannot use their preferences to make a choice from.

To help visualize this, let's look at preferences in terms of a **directed graph**.

Definition 3.4: Directed Graph. In mathematics, a **directed graph** is a set of vertices and edges that have a direction.

Here is how we can use a directed graph to visualize a preference relation. Suppose we have the following complete and transitive preference relation on the set $\{a, b, c\}$.

$$a \succsim b, a \succsim c, b \succsim c, a \succsim a, b \succsim b, c \succsim c$$

Below, I have created a directed graph from these preferences where there is an arrow pointing from one letter to another if the first is preferred to the second. For instance, there is an arrow pointing from a to b since $a \succsim b$. We can leave off arrows from each letter to themselves.

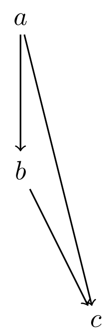


Figure 3.1: A Complete and Transitive Relation on $\{a, b, c\}$

> **Info 3.4: Directed Graph from Preference Relation.** To create a directed graph from a preference relation, create a vertex for each object/bundle. If, for two *distinct* bundles x and y, $x \succsim y$ then draw a directed edge from x to y.

Look how such a complete and transitive relation has a natural ordering. Things higher up in this visualization, like a, are better than everything lower. Even when we get more objects and some indifferences, the same kind of shape appears again. Let's look at the directed graph the following complete and transitive relation (it is much easier to visualize with the graph).

$$a \succsim b, b \succsim a, a \succsim c, a \succsim d, b \succsim c, b \succsim d, c \succsim d, d \succsim c, a \succsim e,$$
$$b \succsim e, c \succsim e, d \succsim e, a \succsim a, b \succsim b, c \succsim c, d \succsim d, e \succsim e$$

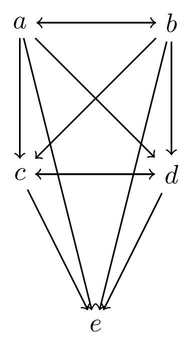


Figure 3.2: A Complete and Transitive Relation on $\{a, b, c, d, e\}$

One nice property of such a shape is that, whatever set of objects a consumer might have to choose from, there is some object in that subset that is at least as good as everything else in that set. In the above example, for instance, a and b are at least as good as everything in $\{a, b, c, d, e\}$. b is at least as good as everything in $\{b, d, e\}$. Whatever set we pick, there is at least one object like that. For whatever set the consumer might be asked to choose from, there is at least one **best** object – something they would be happy to choose.

Definition 3.5: Best. x is **best** from some set B (that includes x) if $x \succsim y$ for every y in B.

We sometimes denote the set of **best** outcomes/options from a set as $C(B)$. This is called the **choice function**. It is the set of best things from a set. Or rather, the things the consumer would be willing to choose. As an example, for the preferences graphed above, $C(a, b, c, d, e) = \{a, b\}$ and $C(b, c, e) = \{b\}$.

What if we make the relation incomplete by removing $a \succsim c$, what is best from the set of $\{a, c\}$? There is nothing as good as everything else, because the consumer has no idea how to compare a and c. That is $C(\{a, c\}) = \emptyset$. So, somewhat trivially, when a relation is not complete, there are menus of objects that the consumer cannot choose form. In a less trivial way, this also happens when we make the relation intransitive.

Suppose we have the following complete but **intransitive** preference relation on the set $\{a, b, c\}$.

$$a \succsim b, b \succsim c, c \succsim a, a \succsim a, b \succsim b, c \succsim c$$

.

$$\begin{tikzcd} & a \arrow[dl] & \\ b \arrow[rr] & & c \arrow[ul] \end{tikzcd}$$

Figure 3.3: A Complete but Intransitive Relation on $\{a, b, c\}$

What is best from $\{a, b, c\}$? That is, what is $C(\{a, b, c, \})$? There is nothing at least as good as everything else. a is not at least as good as c, b is not at least as good as a, c is not at least as good as b. What would the consumer choose?!? We have $C(\{a, b, c\}) = \emptyset$. Intransitive preferences create these kinds of cycles (look at the figure again), and when there are cycles, there are sets that the consumer cannot choose from.

3.4 Chain Notation

When preferences are complete and transitive there is a more convenient way to express them. Look again at this preference relation:

$$a \succsim b, b \succsim a, a \succsim c, a \succsim d, b \succsim c, b \succsim d, c \succsim d, d \succsim c, a \succsim e,$$
$$b \succsim e, c \succsim e, d \succsim e, a \succsim a, b \succsim b, c \succsim c, d \succsim d, e \succsim e$$

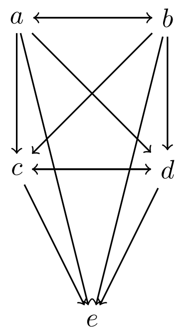


Figure 3.4: A Complete and Transitive Relation on $\{a, b, c, d, e\}$

To summarize the preferences in words, a and b are indifferent, but both of these are strictly better than either c or d which are also indifferent, and these in-turn are both strictly better than e. We can succinctly summarize this by writing:

$$a \sim b \succ c \sim d \succ e$$

. This is what I call **chain notation**.

> **Definition 3.6: Chain Notation.** Chain notation succinctly summarizes a complete and transitive preference relation. In chain notation, only $\succ$ and $\sim$ are used and the "chain" is arranged such that things that come earlier in the chain are at least as good as anything that comes later in the chain.

Consider these preferences:

$$a \succsim b, a \succsim c, b \succsim c, a \succsim a, b \succsim b, c \succsim c$$

The chain notation for these is $a \succ b \succ c$.

> **Info 3.5: Rational Preferences and Chain Notation.** If preferences are complete and transitive, they can always be written in chain notation.

3.5 Indifference Curves and Other Sets

For every object, we can use the preference relation to define a few sets. $\succsim (x)$ is the set of objects that is at least as good as x. $\succ (x)$ is the set of objects that is strictly better than x. $\sim (x)$ is the set of objects indifferent to x.

> **Definition 3.7: Weakly Preferred Set.** The set of points weakly preferred to x is:
>
> $$\succsim (x) = \{y | y \in X, y \succsim x\}$$

Definition 3.8: Strictly Preferred Set. The set of points strictly preferred to x is:

$$\succ (x) = \{y | y \in X, y \succ x\}$$

Definition 3.9: Indifference Curve. The set[a] of points indifferent to x is:

$$\sim (x) = \{y | y \in X, y \sim x\}$$

[a] While $\sim (x)$ is also a set of points like $\succsim (x)$ and $\succ (x)$, we usually call $\sim (x)$ and indifference **curve** rather than an indifference set because for most of the examples used in economics the set has no "thickness". It is just... a curve.

Indifference curves are very important in studying preferences. We call such a set of bundles an "indifference curve". We use indifference curves to visualize preferences. Note: There are many indifference curves. We only sketch a few to get an idea of the "shape" of the preferences.

3.6 Marginal Rates of Substitution and Slope of the Indifference Curve

Definition 3.10: Marginal Rate of Substitution.
The **Marginal Rate of Substitution** measures the slope of an indifference curve at some point. This slope measures, relatively, how much x_2 a consumer would give up to get a little more x_1.

We will often be a little loose about interpreting the MRS and say that is measures, approximately, how much x_2 a consumer will give up to get one more unit of x_1. This is not precisely correct since the MRS is a "local" measurement and really measures the relative amount the consumer will give up of x_2 for an infinitesimally small increase in x_1. However, most importantly, **the MRS measures the willingness to trade-off between good 1 and good 2**.

3.7 Examples of Preferences and Their Indifference Curves

3.7.1 Perfect Substitutes

Definition 3.11: Perfect Substitutes.
Perfect substitutes preferences are preferences where a consumer's willingness to trade off between two goods is always the same. Prefect substitutes preferences have indifference curves that are straight lines.

For example, if a consumer likes apples, but does not care whether an apple is a red apple x_1 or green apple x_2, they are always willing to give up one x_2 to get one x_1. Since the slope of the indifference curve encodes this trade-off the indifference curves for this consumer will be lines with slope of -1.

For other types of **perfect substitutes** preferences, a steep slope indicates a stronger relative preference for x_1. A shallow slope indicates a stronger relative preference for x_2.

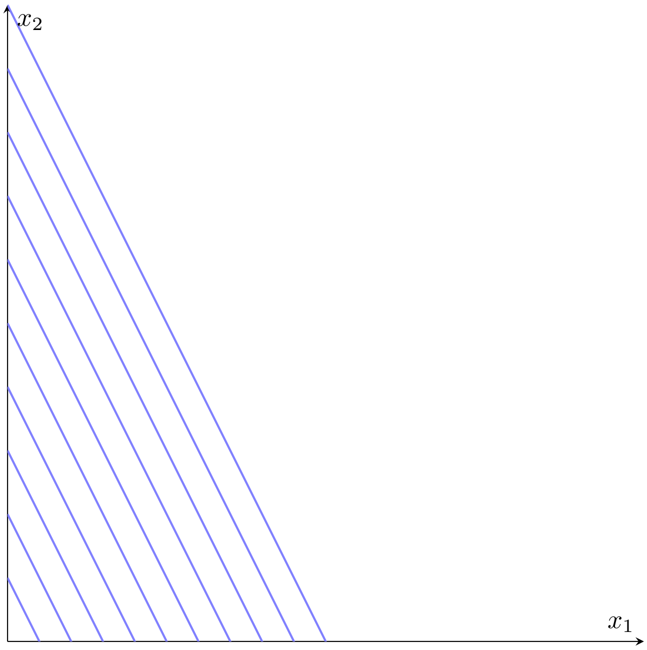


Figure 1: Some indifference curves of a perfect substitutes preference where the consumer will always give up 2 units of x_2 to get 1 unit of x_1.

3.7.2 Perfect Complements

Definition 3.12: Perfect Complements.
Prefect complements represent preferences over goods that must be consumed in fixed proportions. Perfect complements preferences have indifference curves that are L-shaped.

Prefect complements preferences represent situations where one good *cannot* substitute for the other. For example, left and right shoes. No matter how many left shoes you have, they cannot replace a right shoe. You must consume them in 1-to-1 combinations.

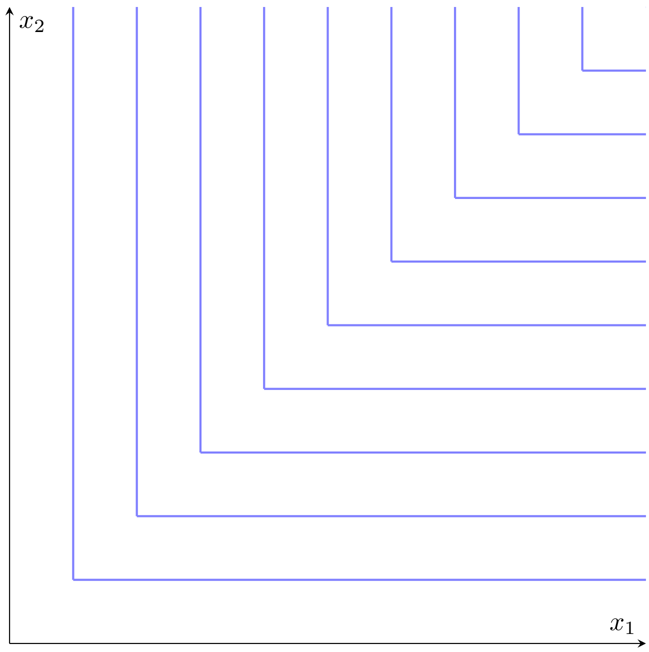


Figure 2: Some indifference curves of a perfect complements preferences where the consumer needs the goods in 1-to-1 combinations.

3.7.3 Bads

Definition 3.13: Bads.
If a consumer would prefer to have *less* of a good, then the good is a bad.
If only one of the two goods is bad, the indifference curves slope upward. If both are bad, the indifference curves slope downward, but preference increases as we move toward the point $(0, 0)$.

For example, if a consumer likes ice cream x_1, but dislikes broccoli x_2 then if you gave them more x_1 and took away x_2 they would be better off, not indifferent. If you give them more x_1 you also have to give them more x_2 to make them indifferent. Thus, the indifference curves slope *upwards*.

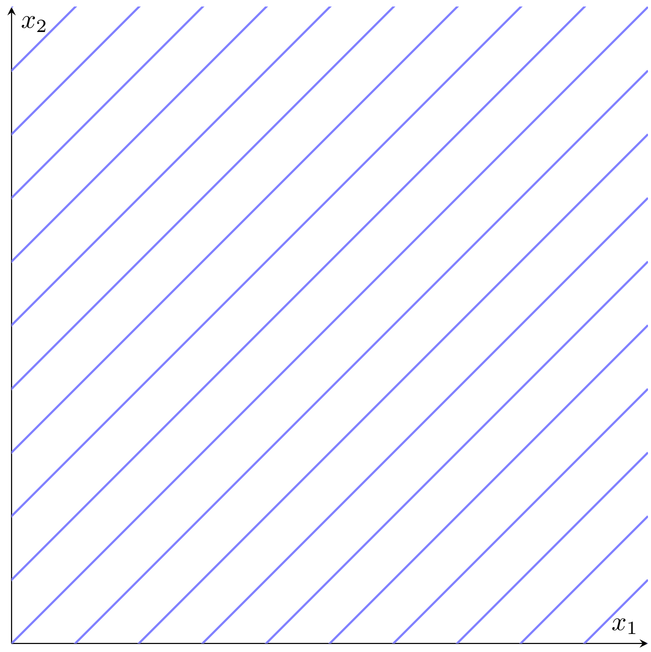


Figure 3: Some indifference curves where x_2 is a *bad*.

3.8 Indifference Curves Cannot Cross

If preferences are rational, there is one thing we can rule out about the shape of indifference curves: **indifference curves cannot cross**.

> **Info 3.6: Indifference Curves Cannot Cross.**
> If $\succsim$ is rational, two distinct indifference curves cannot cross.

The proof of this is remarkably simple and we will discuss it in class, though *you are not responsible for the proof.*

3.9 Key Topics

- Understand how a preference relation is used and defined and how the weak preference relation $\succsim$ can also describe **strict preference** $\succ$ and **indifference** $\sim$.
- Given a weak preference relation, write the strict preference relation as in *Exercise 3.4*.
- Given a weak preference relation, write the indifference relation as in *Exercise 3.5*.
- Understand what properties are needed for a preference relation to be called rational and why those properties are important.
- Given a weak preference relation, determine whether it is complete and transitive as in *Exercise 3.6-3.8*.

- Understand what it means for something to be **best** from a set given a relation and be able to determine what is best from a set given a relation as in *Exercise 3.9*
- Be able to write a preference relation in **chain notation** as in *Exercise 3.3.*
- Understand the definition and use of indifference curves, strictly preferred sets, and weakly preferred sets.
- **Draw** indifference sets, strictly preferred sets, and weakly preferred sets for some described preferences as in *Exercises 3.1-3.2.*
- Understand that the slope of an indifference curve at a bundle is called the **marginal rate of substitution** and that this slope measures a consumer's willingness to trade off between two goods.

4 Utility

A utility function is a way to assign "scores" to bundles, so that better bundles according to $\succsim$ get a higher score. Utility functions allow us to use familiar tools of mathematics to study preferences.

Let's return to our directed graph of a complete and transitive preference relation from the last chapter. Recall that here, things higher up are better than anything lower down. It is always possible to graph preferences this way as long as the preferences are complete and transitive. This time, let's add some numbers to each level of the graph where things higher up get higher numbers.

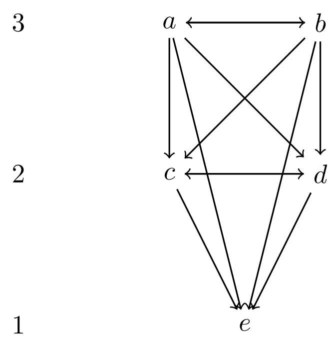


Figure 4.1: A Complete and Transitive Relation on $\{a, b, c, d, e\}$ with Utility

Notice that the number represents "how good" an something is here. a and b get a number 3. They are both indifferent to each other, but strictly better than everything else. c and d get the number 2. They are indifferent to each other but strictly better than e which gets the number 1. We can think of these numbers as "scores" that represent the preferences. In fact, this is precisely what we call a utility function.

4.1 Definition

> **Definition 4.1: Utility Function.** A utility function $U(x)$ represents preferences $\succsim$ when, for every pair of bundles x and y, $U(x) \geq U(y)$ if and only if $x \succsim y$.

Some utility functions that represent the preferences in the example above (in chain notation: $a \sim b \succ c \sim d \succ e$) are $U(a) = 10$, $U(b) = 10$, $U(c) = 5, U(d) = 5, U(e) = 0$ and also $U(a) = 12$, $U(b) = 12$, $U(c) = 10, U(d) = 10, U(e) = -100$.

Note that utility functions simply represent the underlying preference relation $\succsim$ of a consumer. Again, there are many utility functions that can represent the same preferences.

4.2 Examples of Utility Functions

When there is a large number of alternatives, the preference relation itself can be cumbersome to work with. However, a utility function can effectively characterize a preference relation in a succinct way.

Suppose a consumes red apples r and green apples g. They like green apples twice as much as red apples, so they would always give up two red apples in exchange for one green apple. These are **perfect substitutes** preferences.

For combinations of red apples and green apples (r, g) this consumer has preference where, for example:

$$(2,0) \sim (0,1)$$
$$(2,1) \succ (0,1)$$
$$(0,1) \succ (1,0)$$

We can summarize these preferences with the utility function $u(r, g) = r + 2g$.

> **Definition 4.2: Perfect Substitutes.**
> Perfect substitutes preferences can be represented with a utility function of this form: $u(x_1, x_2) = ax_1 + bx_2$

Suppose a consumer only cares about apple pies. An apple pie is made from apples a and crusts c. It takes exactly 2 apples and 1 crust to make a pie.

For combinations of apples and crusts (a, c) this consumer has preference where, for example:

$(2,0) \sim (0,1)$ (Since both make zero pies.)
$(2,1) \sim (2,2)$ (Since both make one pie.)
$(4,2) \succ (2,1)$ (Since the first makes two and the second makes one pie.)

We can summarize these preferences with the utility function $u(r, g) = min\left\{\frac{1}{2}a, c\right\}$.

> **Definition 4.3: Perfect Complements.**
> Perfect complements preferences can be represented with a utility function of this form: $u(x_1, x_2) = min\{ax_1, bx_2\}$

4.3 Transformations

Recall that there are many utility functions that can represent the same preferences. In fact, if we take one utility function, we can get another that represents the same preferences by **transforming** it with a **strictly increasing** function.

> **Definition 4.4: Increasing Transformations.**
> Any strictly increasing function of a utility function represents the same preferences as the original utility function.

For example, suppose: $U(x_1, x_2) = x_1 + x_2$. This represents the perfect substitutes preferences of someone who likes x_1 just as much as x_2. The function $f(z) = z + 100$ is a strictly increasing function. Thus, if we transform the utility function by adding 100, it will represent the same preferences: $\tilde{U}(x_1, x_2) = x_1 + x_2 + 100 = U(x_1, x_2) + 100$.

We could also square the utility and get the same preferences since $f(z) = z^2$ is strictly increasing when z is positive. So, as long as the original utility is always positive, the transformation will represent the same preferences. $\tilde{U}(x_1, x_2) = (x_1 + x_2)^2 = (U(x_1, x_2))^2$.

4.4 MRS from Utility Function

Recall our example above. A consumer eats both red apples r and green apples g. They like green apples twice as much as red apples, so they would *always* give up two red apples in exchange for one green apple. Thus, the **marginal rate of subsitution** or **MRS** for this consumer is *always* -2.

Recall that we can summarize these preferences with the utility function $u(r, g) = r + 2g$. As it turns, out, we can always extract the MRS from the utility function!

First, we need to define the **marginal utility**.

> **Definition 4.5: Marginal Utility.** The Marginal Utility of good i measures how quickly utility increases when we increase that good. It is the partial derivative of the utility function with respect to that good's quantity $mu_i = \frac{\partial u(x_1,x_2)}{\partial x_i}$.

The MRS is the negative of the ratio of marginal utilities:

> **Definition 4.6: The MRS is the (negative of) the Ratio of Marginal Utilities.**
> $MRS = -\frac{mu_1}{mu_2} = -\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}}$

For example, in our example above with $u(r, g) = r + 2g$, here we do not have x_1 and x_2, *but* we know that the consumer would give up 2 g to get 1 r. So, if we treat g as the variable on the vertical axis, that is we treat it like x_2, then the MRS is -2. Let's confirm this by extracting the MRS from the utility function. The marginal utilities are $mu_r = 1$ and $mu_g = 2$. Thus, the MRS is $\frac{mu_g}{mu_r} = -2$ exactly as we expect!

Note that, because two preferences that are the same will have the same indifference curves, they will also have the same MRS.

> **Definition 4.7: Same MRS, Same Preferences.**
> Two utility functions with the same MRS everywhere represent the same preferences.

4.5 More Examples of Utility Functions

4.5.1 Quasi-Linear

> **Definition 4.8: Quasi-Linear.**
> Quasi-Linear preferences can be represented with a utility function of the form: $u(x_1, x_2) = f(x_1) + x_2$

With quasi-linear preferences, the consumer gets tired of one of the two goods. For example, x_1 might be ice cream and x_2 might be money. You get tired of ice cream if you have too much of it, but it is always nice to have more money.

One common quasi-linear utility function is $u(x_1, x_2) = ln(x_1) + x_2$. The marginal rate of substitution for these preferences are $MRS = -\frac{\frac{\partial(ln(x_1)+x_2)}{\partial x_1}}{\frac{\partial(ln(x_1)+x_2)}{\partial x_2}} = -\frac{1}{x_1}$. To interpret this, notice the amount of x_2 the consumer is willing to give up *decreases* as x_1 increases but *does not* depend on the amount of x_2 they have. Using our example above, the amount of money (x_2) the consumer is willing to give up to get more ice cream (x_1) is decreasing the more ice cream they have.

Another example of a quasi-linear utility function is $u(x_1, x_2) = \sqrt{x_1} + 10x_2$. Practice taking the MRS of this function. Notice that it only depends on the amount of x_1.

4.5.2 Cobb-Douglass

Definition 4.9: Cobb-Douglass.
Cobb-Douglass preferences can be represented with a utility function of the form: $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$

With Cobb-Douglass preferences, the consumer consumer gets tired of *both* goods. A general form of these preferences is represented by the utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$. Let's look at the marginal utilities. $mu_1 = \frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta}$, $mu_2 = \frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1}$. Thus, the MRS is $MRS = -\frac{mu_1}{mu_2} = -\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta-1}} = -\frac{\alpha}{\beta}\frac{x_2}{x_1}$. Note that as the ratio of $\frac{x_2}{x_1}$ increases (the consumer has proportionately more x_2) the MRS increases and they are willing to give up more x_2 to get x_1.

4.6 Key Topics

- (*Conceptual.*) Understand how a utility function can be used to *represent* preferences and be able to discuss this in everyday language.
- (*Conceptual.*) Understand what it means for a utility function to be **ordinal**.
- Given a utility function over a small number of objects/bundles determine what preferencse it represents as in *Exercise 4.1*.
- Understand that multiple utility functions can represent the same preferences as in *Exercise 4.2*.
- Given a utility function, determine what is true about the underlying preferences as in *Exercises 4.3-4.6*.
- Given a preference relation, write down a utility function that represents it as in *Exercise 4.7*.
- Given a simple utility function, sketch indifference curves as in *Exercises 4.10*.
- Given a utility function, determine the MRS in general and the MRS at a particular point as in *Exercise 4.8*.

- Given two utility functions, determine whether they represent the same preferences by checking whether they have the same MRS everywhere as in *Exercise 4.9*.

5 Well Behaved Preferences

5.1 Monotonicity

Monotonicity is the assumption that everything is a "good" that is, more of a single good makes the consumer at least as well off and *more of both* makes them strictly better off.

Definition 5.1: Monotonicity.
For two bundles, if one has at least as much of both goods, it is weakly preferred. If one has strictly more of both goods it is strictly preferred. That is for bundles (x_1, x_2) and (y_1, y_2):

1. If $x_1 \geq y_1$ and $x_2 \geq y_2$ then $(x_1, x_2) \succsim (y_1, y_2)$.
2. If both $x_1 > y_1$ and $x_2 > y_2$ then $(x_1, x_2) \succ (y_1, y_2)$.

Monotonicity rules out "bads" where something makes a consumer strictly worse off. In turn, this implied that **indifference curves cannot slope upwards**.

To check if preferences are monotonic from their utility function, we can check if the utility function is monotonic.

Definition 5.2: Monotone. A function $u()$ is said to be **monotonic** when for two bundles (x_1, x_2) and (y_1, y_2):

1. $x_1 \geq y_1$ and $x_2 \geq y_2$ implies $u(x_1, x_2) \geq u(y_1, y_2)$.
2. $x_1 > y_1$ and $x_2 > y_2$ implies $u(x_1, x_2) > u(y_1, y_2)$.

Info 5.1: Preferences are Monotinic when the Utility is Monotonic. If $u(x_1, x_2)$ is monotonic, then the preferences it represents are monotonic.

5.2 Convexity

Convexity is the assumption that mixtures are better than extremes.

Definition 5.3: Convexity.
Preferences are **convex** if for every pair of indifferent bundles $(x_1, x_2) \sim (y_1, y_2)$, a mixture of these bundles is at least as good as the original bundles.

That is, for any $t \in [0, 1]$, the mixture given by $(tx_1 + (1-t)\, y_1, tx_2 + (1-t)\, y_2)$ is at least as good as the endpoints.
$(tx_1 + (1-t)\, y_1, tx_2 + (1-t)\, y_2) \succsim (x_1, x_2)$
$(tx_1 + (1-t)\, y_1, tx_2 + (1-t)\, y_2) \succsim (y_1, y_2)$.

It is useful to think about the shape of convex preferences in terms of their indifference curves. Assuming preferences are monotonic, if preferences are convex, then between any two points on an indifference curve, that curve lies below (or on) a line drawn between those two points.

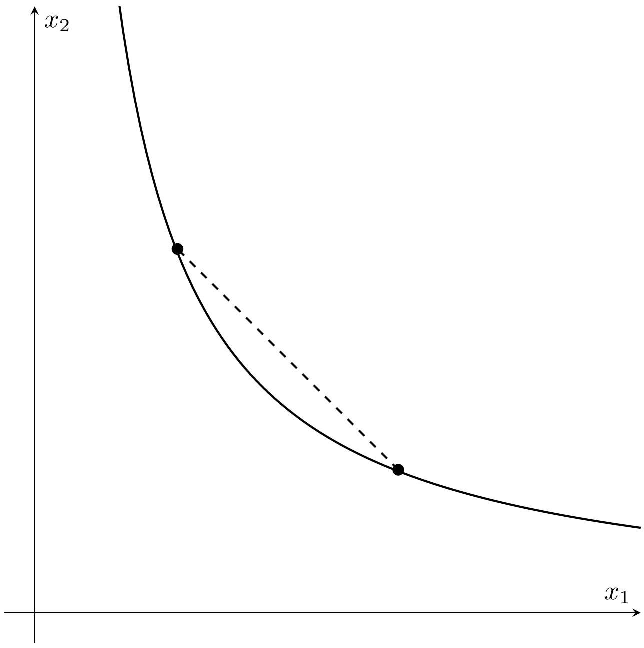


Figure 4: Indifference Curves of a Convex Preference Relation. A line segment (dashed line) has been placed between two points on one of the indifference curves (solid line). Notice that it lies completely above that indifference curve.

5.3 Well Behaved

Definition 5.4: Well Behaved Preferences. When a preference relation is *complete, transitive, monotonic* and *convex* we say it is **well behaved**.

As we will see later in the course, and as their name implies, well behaved preferences are easy to work with.

Info 5.2: Well-Behaved Families of Preferences. Of the families of preferences we have looked at, perfect substitutes, perfect complements, Cobb-Douglass, and quasi-linear preferences, are all *well behaved.*

5.4 Key Topics

- *Conceptual.* Understand the intuitive meaning of monotonicity and know that it rules out upward sloping indifference curves.
- *Conceptual.* Understand the intuitive meaning of convexity and know that it implies that the indifference curves bend "outward".

- Understand the definition of monotonicity and work with preferences represented by a utility function to check whether those preferences are monotonic as in *Exercise 5.2*.
- Understand the definition of a convex combination and be able to find convex combinations of two points as in *Exercise 5.3*.
- Understand the definition of convex preferences and use the notion of convex combinations to test whether preferences are convex for simple examples as in *Exercise 5.1, 5.4*.

6 Constrained Optimization

Imagine yourself hiking on a mountain on a very foggy day. How would you know you were at the peak of the mountain? You might ask yourself "From where I am standing, can I go up?" If the answer is yes, you definitely are *not* at the peak. If the slope is not zero in every direction, then there is *some* direction you can move and go up.

Now, suppose that we want to maximize a utility function $u(x_1, x_2)$ and we can choose any x_1, x_2 we want, , things are not much different. We have to look for a place where the slope of $u(x_1, x_2)$ is zero in every direction.

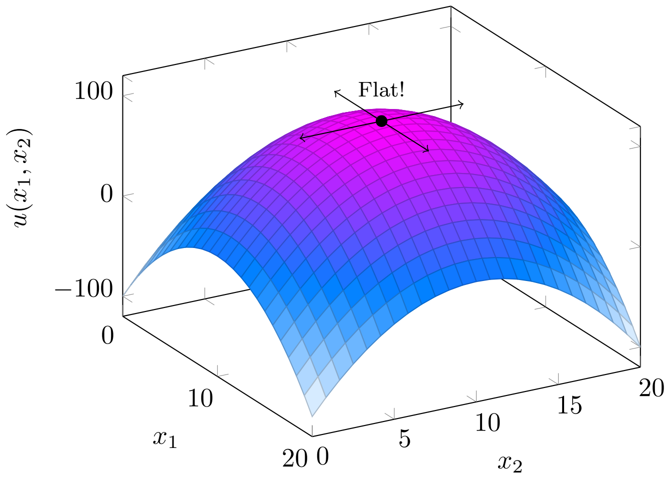


Figure 6.1: Slope is Zero in All Directions!

This is known as *unconstrained optimization*. For example, suppose we want to maximize the function $u(x_1, x_2) = 100 - (x_1 - 10)^2 - (x_2 - 10)^2$ and we are unconstrained. We need to find where the slope in the direction of x_1 is zero and the slope of the function in the direction of x_2 is zero. That is,

$$\frac{\partial\left(u\left(x_1, x_2\right)\right)}{\partial x_1} = 0, \frac{\partial\left(u\left(x_1, x_2\right)\right)}{\partial x_2} = 0$$

That is, where $-2\left(x_1 - 10\right) = 0$ and $-2\left(x_2 - 10\right) = 0$. Simplifying these, we get $x_1 = 10$ and $x_2 = 10$. In fact, this is exactly the function plotted in Figure 6.1. Unconstrained optimization is discussed further in Appendix section B.

Unfortunately, consumers are not unconstrained. They cannot have *any* bundle of (x_1, x_2), they can only have bundles in their budget set B. That is, they are *constrained* to choose from affordable bundles.

Let's have a look at how adding a constraint complicates things. Suppose want to maximize the utility function $u(x_1, x_2) = 100 - (x_1 - 10)^2 - (x_2 - 10)^2$ subject to the budget constraint $x_1 + x_2 \leq 10$.

Let's look at this function first. The function is plotted in Figure 6.2. As we found above, the global maximum (black dot) occurs where $x_1 = 10$ and $x_2 = 10$, but that violates the constraint since $10 + 10 > 10$. The constraint is is represented by the points "below" the red line. We are not allowed to go past the red line. The maximum within that area occurs at $x_1 = 5$ and $x_2 = 5$ (green dot). In the context of finding an optimal bundle subject to a budget constraint, this point is called the **demand**.

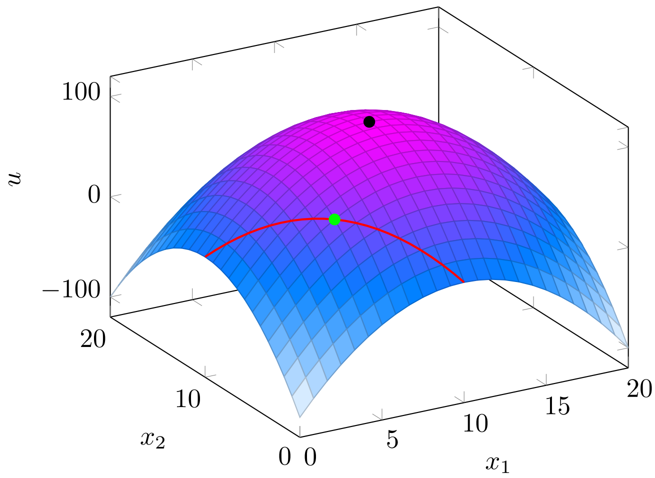
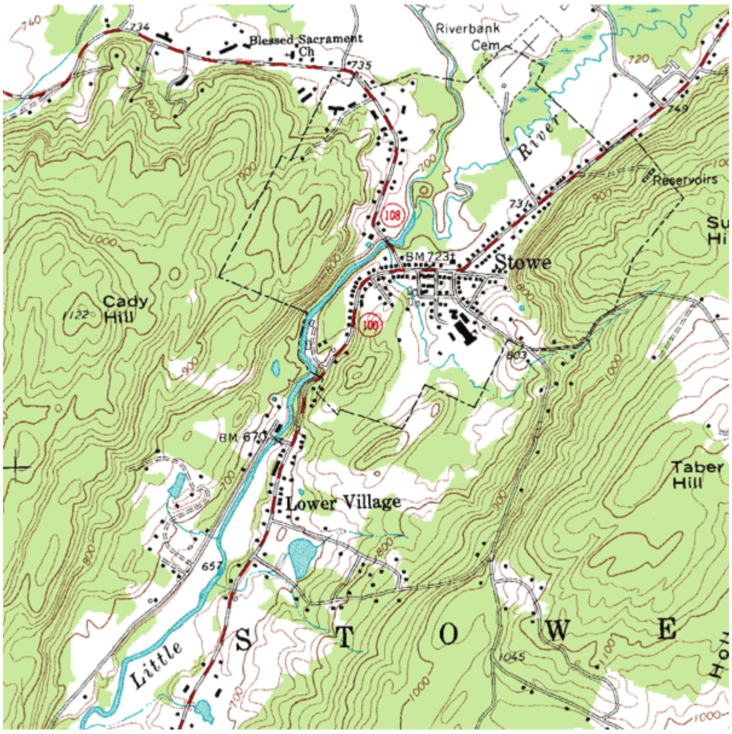


Figure 6.2: Maximizing $100 - (x_1 - 10)^2 - (x_2 - 10)^2$ **subject to** $x_1 + x_2 \leq 10$

How should we formalize the process of finding the constrained optimum? Let's work through a few concepts and return to this example later in the chapter.

6.1 Contours

It can be very useful to think of three-dimensional plots in terms of their contours. Figure 6.3 shows a real-world example of how contours are used on a topographic map, which is a 2d map that includes information about elevation through contour lines. Look at the line labeled "1000" near Cady Hill. This is a line connecting places that all have an elevation of 1000 feet.



Taken from the public domain USGS Digital Raster Graphic file o44072d6.tif for the Stowe, VT quadrangle.

Figure 6.3: A topographic map of Stowe Mountain.

Let's add some contours to our function at an "elevation" of 25, 50, 75, and 99 (right near the peak). In the context of mathematics, this is known as a "contour" plot.

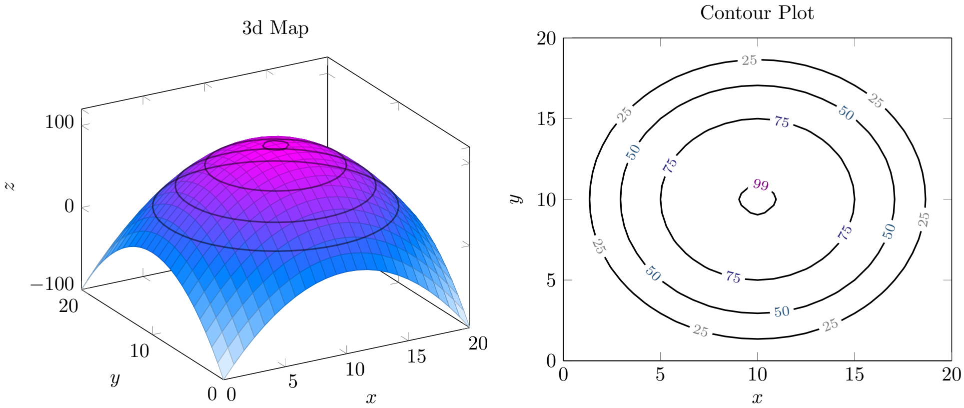


Figure 6.4: A function and its contours.

Note that in the context of a utility function, a contour is a set of bundles that all have the same utility. Thus, they are simply **indifference curves**!

6.2 Monotonicity

Imagine standing at the point $(0, 0)$ on the function plotted in Figure 6.5 below. It is the same as the function plotted in Figure 6.2 but only shown for $x_1 \leq 10$ and $x_2 \leq 10$. If you walk in the northwest direction (increasing x_1 and x_2) the function increases. That is, you are increasing in elevation. In fact, this is true whenever $0 \leq x_1 \leq 10$ and $0 \leq x_2 \leq 10$. Notice how the function always slopes up when moving in the northwest direction *regardless* of where you are. The function is **monotonic**.

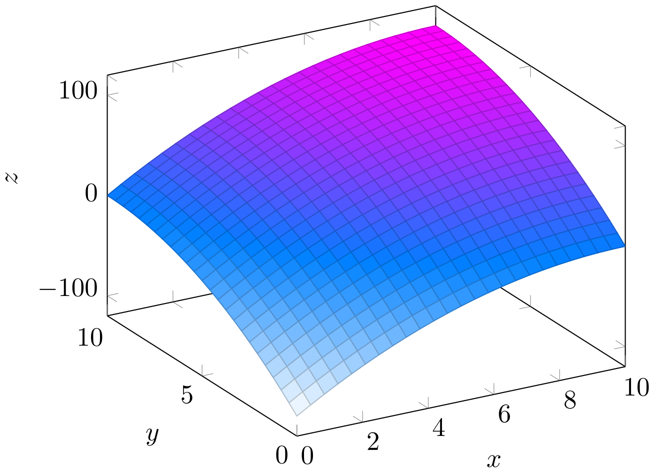


Figure 6.5: 3d Plot of $100 - (x - 10)^2 - (y - 10)^2$ **where** $x \leq 10$ **and** $y \leq 10$**.**

6.3 Three Possibilities for an Optimal Bundle

Let's continue looking at this constrained optimization problem. Now we will look at plot of some contours. That is, the indifference curves of $u(x_1, x_2) = 100 - (x_1 - 10)^2 - (x_2 - 10)^2$ as well as the budget line $x_1 + x_2 = 10$. This is shown in Figure 6.6.

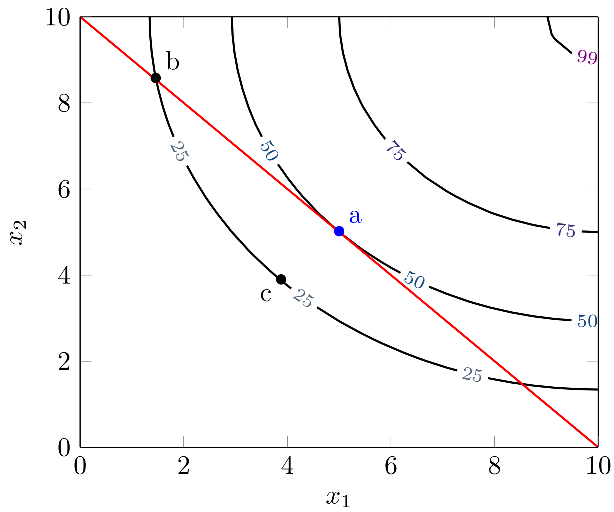


Figure 6.6: Indifference Curves with constraint.

Here, the budget set is the area southwest of the budget line shown in red. The budget line is the "boundary' of the budget set, and the area to the south-west of the budget line is called the interior of the budget set. For example, the points b and a are on the budget line and c is on the interior of the budget set.

First, notice that point c could *never* be optimal. Why? If we are on the interior of the budget set, we could always move up and to the right a little (increasing both x_1 and x_2) and still be at an affordable bundle. *Since the function is* ***monotonic***, the result **must be better!** Here, for uexample, we could move from c to a. Thus, we can see that if a function is monotonic, the optimal bundle (demand) *cannot* be on the interior of the budget set. But it still has to be in the budget set. Thus, **when a utility function is monotonic, the optimal bundle must be on the budget line!**

The point b is on the budget line. Can it be optimal? No, it is on the same contour as c. Because c *cannot* be optimal, *neither* can b. They are indifferent. This shows us that a point like b, which is on an indifference curve that passes through the interior of budget set, can *never* be optimal.

Info 6.1: Indifference Curve Cannot Cross into Interior of Budget Set for Optimal Bundle. The indifference curve through the optimal bundle cannot cross into the interior of the budget set.

What we have seen so far is that whatever bundle is optimal *must* be on the budget line and not on an indifference curve that passes into the interior of the constraint. The only way for this to happen is if the indifference curve at the optimal bundle *just touches* the budget line. See point c for instance. When the indifference curves are smooth, the only way for this to happen is if the indifference curve and the budget line have the same slope.

There are only three possibilities for an optimal bundle. These are enumerated below.

Info 6.2: Three Possibilities for an Optimal Bundle. When the utility function is monotonic, the optimum must meet one of the following three conditions.

1. (**Tangent**) It is at a point where the indifference curve at that point had the same slope as the budget line.
2. (**Touching but not Smooth**) The point is a "non-smooth" point on the indifference curve, but that point just touches the budget line.
3. (**Boundary**) The point is at one of the boundaries of budget line where either $x_1 = 0$ or $x_2 = 0$.

Importantly, if we know the indifference curves are smooth everywhere (we can tell because we will be able to take the derivative of the utility function), then for a bundle to be optimal if it includes some of each good (that is, it's not on the boundary) it must be on a tangent point. The slope of the indifference curve at that optimal bundle is the same as the slope of the budget line. This condition is formalized by the familiar equation: $MRS = -\frac{p_1}{p_2}$.

Definition 6.1: Tangency Condition. For maximizing $u(x_1, x_2)$ subject to $p_1x_1 + p_2x_2 \leq m$, the **tangency condition** is:

$$MRS = -\frac{p_1}{p_2}$$

$$-\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = -\frac{p_1}{p_2}$$

.

The **tangency** condition is sometimes called the **first-order** condition in other places. Another way to interpret the tangency condition is as follows. Normally we have $-\frac{MU_1}{MU_2} = -\frac{p_1}{p_2}$ for the tangency condition. This form of the tangency condition says that the slope of the indifference curve is equal to the slope of the budget line. The slope of the indifference curve is willingness to trade-off between two goods while the slope of the budget line is how the consumer has to trade-off given the prices. If their willingness to trade off is different than how they have to trade off, they are either willing to give up more x_2 than they have to in order to get more x_1 or they are willing to give up more x_1 than they have to in order to get more x_2. In either case, they would **not** be doing something optimal!

Info 6.3: Interpreting the Tangency Condition- Tradeoffs.
The tangency condition ensures that the willingness to trade-off between two goods *is the same as* the way the consumer has to trade-off between the goods given the prices. If these trade-offs *are not the same*, then either they are willing to give up more x_2 than they have to in order to get more x_1 or they are willing to give up more x_1 than they have to in order to get more x_2. Thus, assuming they can trade-off in either direction (they are buying some of both goods), these slopes have to be the same!

But, we can rearrange this to $\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$. Notice how if we divide marginal utility by the price of that good, it tells us the marginal utility of a dollar spent on the good. To see this, suppose the marginal utility of a good is 2, but the good costs \$2 per unit. Then spending \$1 gets half a unit which increases utility by about 1. If we take $\frac{MU}{p}$ we also get 1. Thus, we have the following alternative interpretation of the tangency condition.

Info 6.4: Interpreting the Tangency Condition- Marginal Utility Per Dollar.
The tangency condition ensures that the marginal unit per dollar spent on both goods it the same.

6.4 Examples

6.4.1 Cobb Douglass:

Suppose a consumer's utility function is $u(x_1, x_2) = x_1 x_2$ and their budget is defined by $p_1 = 2, p_2 = 1, m = 20$.
Since this a utility function we can take derivatives of, we start with the tangency condition:
The MRS is $MRS = -\frac{\frac{\partial(x_1 x_2)}{\partial x_1}}{\frac{\partial(x_1 x_2)}{\partial x_2}} = -\frac{x_2}{x_1}$.
Thus the tangency condition is $-\frac{x_2}{x_1} = -\frac{p_1}{p_2}$.
Plugging in the prices: $-\frac{x_2}{x_1} = -\frac{2}{1}$ or $2x_1 = x_2$.
We also know the optimal bundle occurs **on the budget line** $2x_1 + x_2 = 20$. This is two equations and two unknowns. We can solve the tangency condition and the budget condition together to get. $x_1^* = 5$ and $x_2^* = 10$. Thus, the optimal bundle (or the **demand**) is $(5, 10)$.

6.4.2 Perfect Substitutes

Suppose a consumer's utility function $u(x_1, x_2) = 2x_1 + x_2$ with budget $1x_1 + 1x_2 = 10$. That is $p_1 = 1$,$p_2 = 1$,$m = 10$. The MRS is $MRS = -\frac{\frac{\partial(x_1 x_2)}{\partial x_1}}{\frac{\partial(x_1 x_2)}{\partial x_2}} = -\frac{2}{1} = -2$. Thus the tangency condition is $-2 = -1$. Since this can never happen, the only possible solution is one on the boundary. That is, it does *not* contain some of both goods.

If the consumer only buys x_1, they can get the bundle $\left(\frac{m}{p_1}, 0\right) = (10, 0)$ which has utility 20. If they only buy x_2, they can get $\left(0, \frac{m}{p_2}\right) = (0, 10)$ which has utility 10.

Thus, the optimal bundle is $(10, 0)$.

Notice that in this problem, if the budget was instead $p_1 = 2, p_2 = 1, m = 10$ or $2x_1 + 1x_2 = 10$ then we would have gotten $-\frac{2}{1} = -\frac{2}{1}$ for the tangency condition. In this case, the indifference curves have the same slope as the budget line. As long as the consumer spends all of their money, any bundle is optimal. This is because the tradeoff they are willing to make is *exactly the same* as the trade-off that the market asks them to make to stay affordable. All of the bundles such that: $p_1 x_1 + p_2 x_2 = m$ are optimal.

6.4.3 Perfect Complements (Left, Right Shoes)

Suppose a consumer buys left x_1 and right x_2 shoes. Their utility function is $u(x_1, x_2) = min\{x_1, x_2\}$ and suppose $p_1 = 2$,$p_2 = 1$,$m = 15$. The budget line is $2x_1 + x_2 = 15$. We still know the budget condition must be true at the optimal. But we cannot take derivatives of

this utility function. There is no MRS. What is the other condition that takes the place of the tangency condition?
The only possible place an indifference curve could just touch the budget line without crossing into it is at the kink points of the indifference curves. Thus, the equation for the kink points, or what I call the "**no waste condition**" serves the place of our tangency condition in the previous problems. The no waste condition in this problem is: $x_1 = x_2$. Solving the no waste condition along with the budget gets us: $x_1 = 5$ and $x_2 = 5$.

6.4.4 Perfect Complements (2 Apples, 1 Crust)

Suppose a consumer makes pies using 2 apples x_1 and 1 crust x_2 for every pie. Their utility function is $u(x_1, x_2) = min\left\{\frac{1}{2}x_1, x_2\right\}$ and suppose $p_1 = 2$,$p_2 = 1$,$m = 15$. The budget line is $2x_1 + x_2 = 15$.

Again, the budget condition must be true at the optimum: $2x_1 + x_2 = 15$. And the no waste condition here is $\frac{1}{2}x_1 = x_2$. Solving these together, we get $x_1 = 6$, $x_2 = 3$.

6.5 Key Topics

- *Conceptual* Understand that, when utility is monotonic, the optimal bundle cannot be on an indifference curve that crosses into the budget set.
- Understand the three possibilities this creates for how an optimal bundle can occur according to Info Box 6.2.
- *Conceptual* Know what the tangency condition is and how to interpret it, both in terms of trade-offs **and** marginal utility per dollar. *Exercises*
- Use the concepts from this chapter to solve utility maximization problems as in *Exercises 6.1-6.8*

7 Demand

7.1 Marshallian Demand

In the examples in the previous chapter, we solved for an optimal bundle given specific prices and income. This is the bundle a consumer "demands" given the prices and their income. We often refer to this simply as their **demand**.

However, we can also solve for demand while leaving these parameters unspecified. The amount they demand can depend on both prices and income. Thus, when we do this, we get a function that can involve p_1, p_2 and m. This is called their **Marshallian Demand**.

> **Definition 7.1: Marshallian Demand.** The amount of a good a consumer demands as a function of p_1, p_2 and m is known as their **Marshallian Demand** for that good. Marshallian demand for good 1 is denoted $x_1^*(p_1, p_2, m)$. and Marshallian demand for good 2 is denoted $x_2^*(p_1, p_2, m)$.

Let's look at an example of the Cobb-Douglass utility: $U(x_1, x_2) = x_1 x_2$. The tangency condition is $-\frac{x_2}{x_1} = -\frac{p_1}{p_2}$ which simplifies to $p_1 x_1 = p_2 x_2$. Notice how the tangency condition says "spend the same amount on both goods". Plugging this into the budget constraint $p_1 x_1 + p_2 x_2 = m$ we get $p_1 x_1 + p_1 x_1 = m$ or $x_1^*(p_1, p_2, m) = \frac{\frac{1}{2}m}{p_1}$ and $x_2^*(p_1, p_2, m) = \frac{\frac{1}{2}m}{p_2}$. Thus, for the Cobb-Douglass utility function $u(x_1, x_2) = x_1 x_2$, the Marshallian demand involves the consumer spending half of their income on good 1 and half of their income on good 2.

Solving for the Marshallian demands allows us to see how demand changes when prices or income changes. We will explore that in this chapter.

7.2 Changes in Income

We will start with the question of "How does demand change with income?" There are two possibilities.

> **Definition 7.2: Normal Good.** A good for which *demand* ***increases*** when *income* ***increases*** is called **normal**.

> **Definition 7.3: Inferior Good.** A good for which *demand* **decreases** when *income* ***increase***s is called **inferior**.

For example, previously, we found that the Marshallian demand for a consumer with utility function $u(x_1, x_2) = x_1 x_2$ is $x_1^*(p_1, p_2, m) = \frac{\frac{1}{2}m}{p_1}$.

Since m is only in the numerator, demand must be increasing as income increases. Because of this the good is normal. Another way to see this is to take the derivative of x_1^* with respect to m. If we do that, we get:

$$\frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial m} = \frac{\frac{1}{2}}{p_1}$$

Since this is a positive number regardless of what p_1 is, it tells us that x_1 increases as m increases.

7.2.1 Engel Curve

The Engel Curve is the relationship between income and a single good.

Definition 7.4: Engel Curve. The **Engel Curve** is a plot of demand for some good x_i^* as income *changes* but prices *remain fixed* with m placed on the vertical axis and x_i^* on the horizontal axis.

Note that for the Engel Curve, we plot m on the vertical axis and the good on the horizontal axis. This may seem awkward at first - it's just a weird tradition in economics! We can think of the plot as telling us the amount of income a consumer would need to have to demand some amount of that good given some prices.

For instance, suppose the Marshallian demand for good 1 is $x_1^*(p_1, p_2, m) = \frac{\frac{1}{4}m}{p_1}$ and we want to look the the Engel Curve for when $p_1 = 1, p_2 = 1$. In that case, we get $x_1^*(1, 1, m) = \frac{1}{4}m$. Since we need to put m on the vertical axis, it is easier to isolate m in this equation. Doing that we get: $m = 4x_1$. If we ask how much income the consumer would need so that they buy 10 units of x_1 we get $m = 4\,(10) = 40$. If the consumer has \$40, they would be able to buy 10 units of the good.

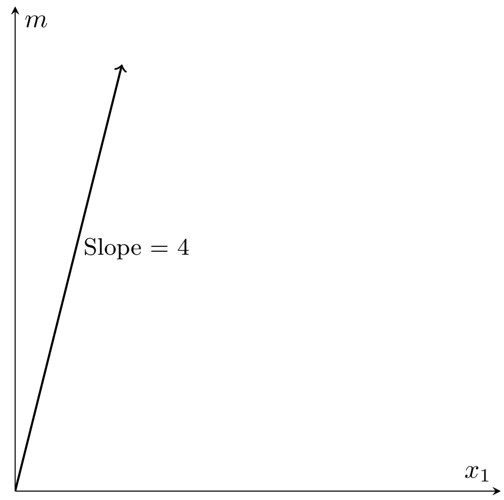


Figure 5: Engel curve for Marshalian demand $x_1^*(p_1, p_2, m) = \frac{\frac{1}{4}m}{p_1}$ when $p_1 = 1$ and $p_2 = 1$.

When demand is normal, the relationship between demand and income is positive and the Engel Curve will have a positive slope. On the other hand, if the good is inferior, demand decreases as income increases, and this is reflected in a "backward bend" of the Engel Curve. This is demonstrated in the plot below.

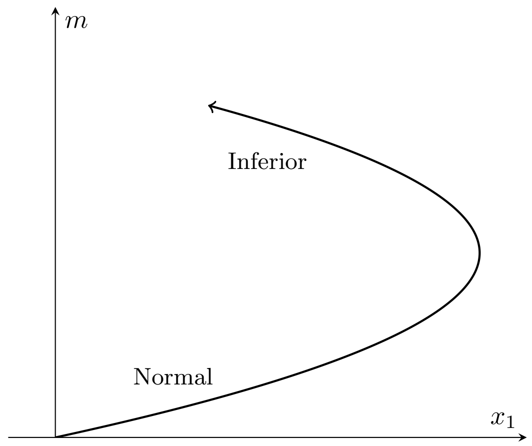


Figure 6: Engel curve for a good that is normal for low income and inferior for high income.

7.2.2 Never Always Inferior.

An *inferior* good is one for which demand decreases as income increases. But, to decrease, the demand for the good *must* be non-zero. That means that at some point it must have increased. Because of this, **a good can never be always inferior**. To be inferior, a good must start out as normal and *become inferior*.

> **Info 7.1: Not Always Inferior.** A good cannot be inferior for every level of income m.

The plot of an Engel curve for x_1 where x_1 starts as normal but becomes inferior is plotted in Figure 7.2.1. Notice how the curve bends backward as x_1 becomes inferior.

7.2.3 Example: Perfect Complements

Suppose that the utility is $U(x_1, x_2) = min\{x_1, x_2\}$ and prices are: $p_1 = 2$, $p_2 = 1$. Let's solve for demand and then plot the income offer and Engle Curve for x_1.

At the optimum, we know $x_1 = x_2$ (this is the "no waste condition"). Since the budget constraint is $2x_1 + 1x_2 = m$ we have two equations and two unknowns. Solving these together, we get demand $x_1 = \frac{m}{3}$, $x_2 = \frac{m}{3}$.

To get the Engel Curve for x_1, solve for m in the demand for x_1. We can rearrange the demand for x_1 to get $m = 3x_1$. This is the Engel curve for x_1. It is a line with slope of 3. This is plotted below.

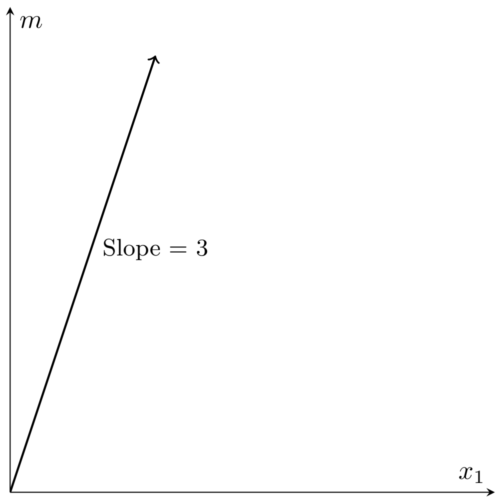


Figure 7: Engel Curve of x_1 for $min\{x_1, x_2\}$ with $p_1 = 2$ and $p_2 = 1$.

7.3 Changes in "Own" Price

We now look at what happens to demand for a good when the price of that good changes. There are two possibilities: Ordinary good and Giffen good.

Definition 7.5: Ordinary Good. A good for which *demand decreased* when its *price increase* is called **ordinary**.

Definition 7.6: Giffen Good. A good for which *demand increases* when its *price increase* is called **ordinary**.

This second possibility seems strange, but it is possible given our general model of preferences and choice. Interestingly, a Giffen good must be inferior. When price of the good increases, the consumer will naturally trade off to buy other things. But whatever amount of the good they continue to buy will now be more expensive. This makes their income effectively lower– it does not buy as much. If a good is inferior, this can lead to an increase in the demand for that good. If this effect overwhelms the decrease in demand due to the consumers trading-off to other goods, the net effect *can* be positive. However, in practice, such goods are hard to find, and we will not study them extensively in this class.

Info 7.2: A Giffen good Must be Inferior. Every Giffen good is also inferior.

7.3.1 Plotting the Demand Curve

The demand for a good is $x_i(p_1, p_2, m)$ that is, the optimal amount that a consumer chooses given the prices and income. When we talk about "plotting" the demand curve of x_1 we usually mean holding p_2 and m fixed and plotting how the demand for x_1 changes as p_1 changes. For this, we usually put p_1 on the vertical axis and x_1 on the horizontal axis. Technically what we are plotting is the **inverse demand curve**.

Definition 7.7: Inverse Demand Curve. The **inverse demand curve** is a plot of demand for some good x_i^* as its price p_i *changes* but the other prices and income *remain fixed*. For this plot, p_i is placed on the vertical axis and x_i^* on the horizontal axis.

For example, suppose that the demand for x_1 is:

$$x_1 = \frac{\frac{1}{2}m}{p_1}$$

Let's plug in an income $m = 10$ and hold that fixed. We get $x_1 = \frac{5}{p_1}$. To plot this with p_1 on the vertical axis, we need to solve for p_1. When we do this, we get $p_1 = \frac{5}{x_1}$ this is the **inverse demand**. This is plotted below.

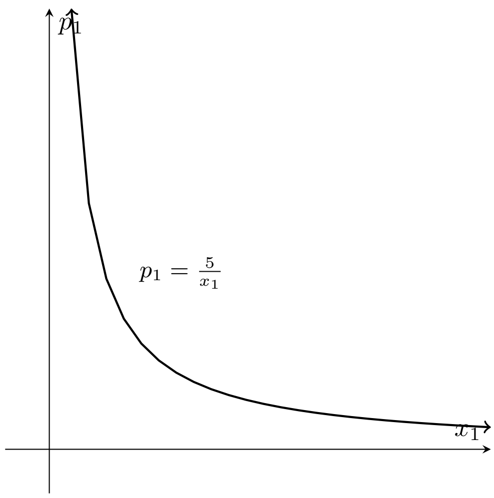


Figure 8: Plotting the inverse demand for $x_1 = \frac{5}{p_1}$.

Info 7.3: Interpreting the Inverse Demand. The inverse demand tells us the price p_i that would be responsible for the consumer buying some amount x_i of the good.

7.4 Changes in "Other" Price

So far we have looked at what happens to a good when we change income and it's own price. We might also be interested in how demand changes for a good when there is a change in the price of another good.

Definition 7.8: Substitutes. If a good's *demand increases* when the *price of another good increase* is called a **substitute** for that other good.

Definition 7.9: Complements. If a good's *demand decreases* when the *price of another good increase* is called a **complement** for that other good.

Definition 7.10: Neither. If a good's *demand does not change* when the *price of another good increase* then their are neither substitutes nor complements.

7.4.1 Perfect Complements Example

$u = min\{x_1, x_2\}\, has$ demand $x_1 = \frac{m}{p_1+p_2}$ and $x_2 = \frac{m}{p_1+p_2}$. For both goods, as you increase the price of the other good, the demand goes down. They act as complements for each other (*hopefully this is not a surprise*).

7.4.2 Perfect Substitutes Example

$u = x_1 + x_2$ has demand $x_1 = \frac{m}{p_1}$ $x_2 = 0$ if $p_1 < p_2$ and $x_1 = 0$ $x_2 = \frac{m}{p_2}$ if $p_1 > p_2$. Let's look at the change in p_1. If $p_1 < p_2$ and p_1 increases, then if it increases enough to that $p_1 > p_2$ the demand for x_2 increases from 0 to $x_2 = \frac{m}{p_2}$. The same happens when we consider the effect of p_2 on demand for x_1. So, as long as the change in a price has any effect on the demand for another good, the goods act as substitutes for each other.

7.5 Key Topics

- *Conceptual* Understand what it means for demand to be normal, inferior, ordinary, Giffen.
- *Conceptual* Understand what it means for a good to be a complement or a substitute for another in terms of how demand reacts to changes in the other good's price.
- Using the Marshallian demand, be able to determine when goods are normal, inferior, ordinary, Giffen, complements, or substitutes (or neither) as in *Exercises 7.2,7.6,7.9*
- Be able to find the Marshallian demand for a good given a utility function as in *Exercises 7.1,7.5,7.7,7.8*
- Be able to plot an Engle curve from a demand function as in *Exercise 7.3*
- Be able to find the inverse demand function for a good and plot the inverse demand as in *Exercise 7.4*

8 Slutsky Decomposition

As previewed by our discussion of Giffen goods above, when a price changes, there are two reasons demand for a good can change. Suppose that the price of a good increases. First, that price increase can induce the consumer to *substitute* away from the now more expensive good and, therefore, to be persuaded to buy other goods. Second, the price increase reduces the effective value of the income of the consumers. This can further change demand depending on if the good is normal or inferior.

Decomposes the change in demand for a good into two parts:

> **Definition 8.1: Substitution Effect.** When price increases, a consumer will substitute into buying more of other goods which are now relatively less expensive. This is the **substitution effect**.

> **Definition 8.2: Substitution Effect.** When the price of a good goes up, a consumer's effective income decreases, this can lead them to buy less/more of a good depending on whether it is normal/inferior. This is the **income effect**.

> **Definition 8.3: Law of Demand.** For a change in the price of good i the substitution effect (on good i) will always lead to a decrease or no change in demand x_i. That is, the substitution effect is always "negative".

The law of demand leads to three possible combinations:

- **Ordinary/Normal**: *If price increases, both effects decrease demand.*
- **Ordinary/Inferior**: *Substitution decreases demand (it always does by the law of demand) and income effect increases demand, but not enough to overcome the decrease due to substitution.*
- **Giffen/Inferior**: *Substitution decreases demand (it always does) and income effect increases demand, but not enough to overcome the decrease due to substitution.*

8.1 The Slutsky Decomposition.

The **Slutsky decomposition** is sort of a thought experiment that allows us to measure the substitution effect and the income effect.

Suppose that the price of good 1 increases from p_1 to p_1'. The budget line changes from the budget $p_1x_1 + p_2x_2 = m$ to $p_1'x_1 + p_2x_2 = m$. When the budget changes, demand for x_1 will change as well.

> **Definition 8.4: Total Effect.** The **total effect** of this change on demand is the difference between demand after a price change and before a price change. For example, if the price of good 1 changes from p_1 to p_1', the total effect is
> $x_1^*(p_1, p_2, m) - x_1^*(p_1', p_2, m)$

How can we decompose the total effect into the substitution and income effect?

To isolate the substitution effect, we need to know what the consumer would choose if the price had changed, but demand could not change due to income. One way to measure this is to imagine what they would do if the price changes, but their income was changed so that, at the new prices, they could still afford the old bundle. Whatever change their demand has after this price change but with the extra income could not be due to income effect! We have compensated their income to be just as valuable after the price change. The only thing left is the substitution effect.

So, to find the substitution effect, we first calculate the *compensated income*. This is the cost of the bundle demanded before the price change but under the new prices.

> **Definition 8.5: Compensated Income.** The **compensated income** $\tilde{m}$ is the amount of money needed to buy the old bundle at the new prices.
> For example, if the price of good 1 changes from p_1 to p_1', the compensated income is:
> $\tilde{m} = p_1' x_1^* (p_1, p_2, m) + p_2 x_2^* (p_1, p_2, m)$

Now we construct a new budget with this compensated income ($\tilde{m}$) and ask: What does the consumer choose on this budget?

However, the bundle differs from their original bundle before the price change cannot be due to the income effect. We have compensated their income in this thought experiment so that it is enough to buy their old bundle. That wipes out the income effect and leaves only the substitution effect.

> **Definition 8.6: Substitution Effect.** The **substitution effect** is the difference between the demand under the new prices but with compensated income and the original demand.
> For example, if the price of good 1 changes from p_1 to p_1', the compensated income is:
> $x_1^* (p_1', p_2, \tilde{m}) - x_1^* (p_1, p_2, m)$

Since the total effect is the substitution effect plus the income effect, we can now measure the income effect with what is leftover.

> **Definition 8.7: Income Effect.** The **income effect** is the total effect minus the substitution effect.

8.2 Example Problem

Suppose $u = x_1 x_2$. Demand is $x_1^* = \frac{\frac{1}{2}m}{p_1}, x_2^* = \frac{\frac{1}{2}m}{p_2}$. Suppose $p_1 = 1$, $p_2 = 2$, $m = 10$. The Optimal Bundle (original prices): is $x_1^* = \frac{\frac{1}{2}10}{1} = 5, x_2^* = \frac{\frac{1}{2}10}{2} = 2.5$.

Now suppose price of good 1 changes to $p_1' = 2$. The new optimal bundle is $x_1^* = \frac{\frac{1}{2}10}{2} = 2.5, x_2^* = \frac{\frac{1}{2}10}{2} = 2.5$. The total effect is $(2.5 - 5) = -2.5$ That is, demand decreases by 2.5 due to the substitution effect.

Let's calculate the compensated income. That is, the income needed to afford old bundle at the new prices. The old bundle is $(5, 2.5)$. The cost of this bundle under the new prices: $p_1 = 2$, $p_2 = 2$ is the *compensated income*. This is $\tilde{m} = 5\,(2) + 2.5\,(2) = 15$.

We now construct a budget that has the new prices with the compensated income: $p_1 = 2, p_2 = 2, m = 15$. What does the consumer actually demand with this budget? $x_1^*\,(2, 2, 15) = \frac{\frac{1}{2}15}{2} = 3.75$.

With this we can calculate the substitution effect. That is the demand under the thought experiment above minus the original demand: $3.75 - 5 = -1.25$. Thus, demand decreases by 1.25 due to the substitution effect.

This leaves the Income Effect: (Total Effect-Substitution): $(-2.5) - (-1.25) = -1.25$. Thus, demand decreases by 1.25 due to the income effect.

8.3 Key Topics

- *Conceptual* Understand how the total effect of a change in demand is composed of two effects: the substitution and income effect and what is responsible for these effects.
- *Conceptual* Understand that the **law of demand** says the substitution is always negative and what this means.
- Use the Slutsky decomposition to measure the substitution and income effect as in *Exercises 8.1,8.2*

9 Buying and Selling

9.1 Income to Endowments

Until this point, our consumers had income in terms of money. For example, an income of $m = \$10$. Now we will think of the consumers as having an "endowment" of goods to start with rather than money. An endowment is a bundle of goods denoted (w_1, w_2). For example, if x_1 represents apples and x_2 represents crusts, then the apple farmer might have the endowment $w_1 = 10, w_2 = 0$. This would be an endowment of 10 apples and zero crusts. A Baker might have an endowment of 5 crusts and zero apples $w_1 = 0, w_2 = 5$.

Definition 9.1: Endowments. The **endowment** (w_1, w_2) is the bundle of goods the consumer starts with in place of an income of money.

When we move from income to endowments, the new budget condition requires that the cost of the chosen bundles be less than or equal to the value of the endowment: $p_1x_1 + p_2x_2 \leq p_1w_1 + p_2w_2$. The new budget equation is: $p_1x_1 + p_2x_2 = p_1w_1 + p_2w_2$.

Definition 9.2: Budget Equation with Endowments. When consumers have an endowment of goods instead of money, their budget line is given by the budget equation:

$$p_1x_1 + p_2x_2 = p_1w_1 + p_2w_2$$

Notice how the effective income of the consumer, given by $p_1w_1 + p_2w_2$, reacts to changes in price. This is an important distinction from when income was given in terms of an amount of money.

As before, we can find the slope of the budget equation, which is still $-\frac{p_1}{p_2}$ and the intercepts. The x_1 intercept (the amount of x_1 afford if I only buy x_1) is $w_1 + \frac{p_2w_2}{p_1}$ and the x_2 intercept is $\frac{p_1w_1}{p_2} + w_2$. Look carefully at these intercepts and see if you can give them an economic interpretation.

9.2 Gross Demand vs. Net Demand

In this model, we distinguish between what a consumer demands, the gross demand: x_i and the difference between their demand and what they started with, the net demand: $x_i - w_i$.

Definition 9.3: Gross Demand. The **Gross Demand** is the amount of a good a consumer chooses x_i.

Definition 9.4: Gross Demand. The **Net Demand** is difference between gross demand and endowment $x_i - w_i$.

When net demand is positive, we say that the consumer is a net buyer of that good. When net demand is negative, we say that they are a net seller of that good.

Definition 9.5: Net Buyer. A consumer is a **net buyer** of good i if they demand more than their endowment $x_i > w_i$

Definition 9.6: Net Seller. A consumer is a **net seller** of good i if they demand less than their endowment $x_i < w_i$

We can also write the budget equation in terms of net demand by rearranging things: $p_1(x_1 - w_1) - p_2(w_2 - x_2) = 0$. In this form of the budget equation, it makes clear that a consumer must have budget balance in terms of net demand. This also shows that if a consumer is a buyer of one good, they are a seller of the other.

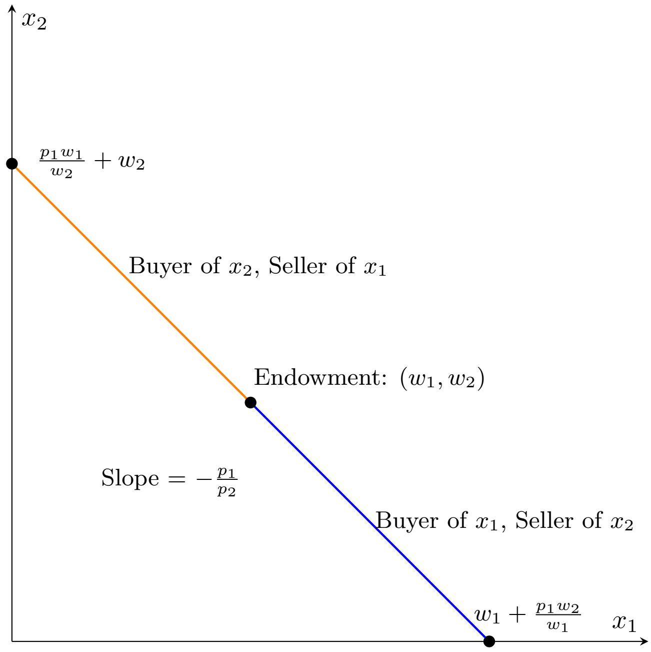


Figure 9: The budget line. The orange part of the budget line above the endowment is where the consumer is a buyer of x_2 and a seller of x_1. The blue part below the endowment is where the consumer is a buyer of x_1 and seller of x_2.

9.3 Drawing the Budget Line and Changes to Price

The budget line always passes through the endowment (w_1, w_2). If prices change, the slope changes, and the budget pivots through this point.

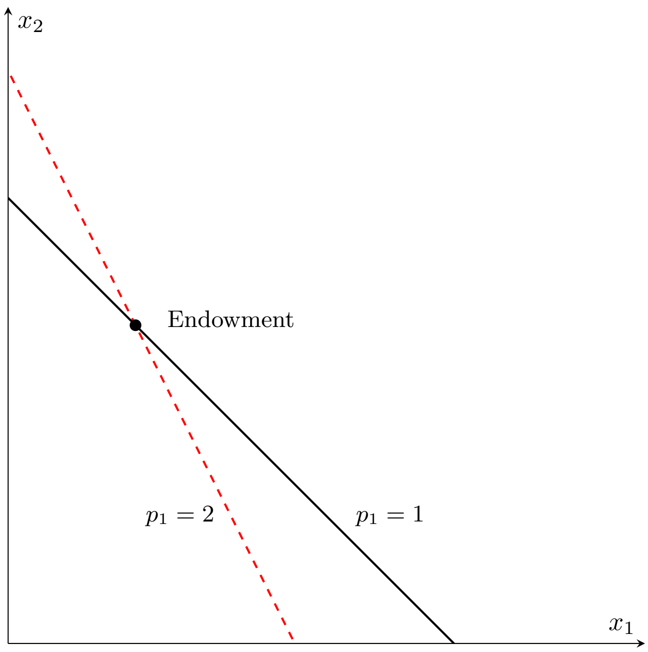


Figure 10: An example of how a change in p_1 affects the budget equation with endowment $(w_1, w_2) = (2, 5)$.

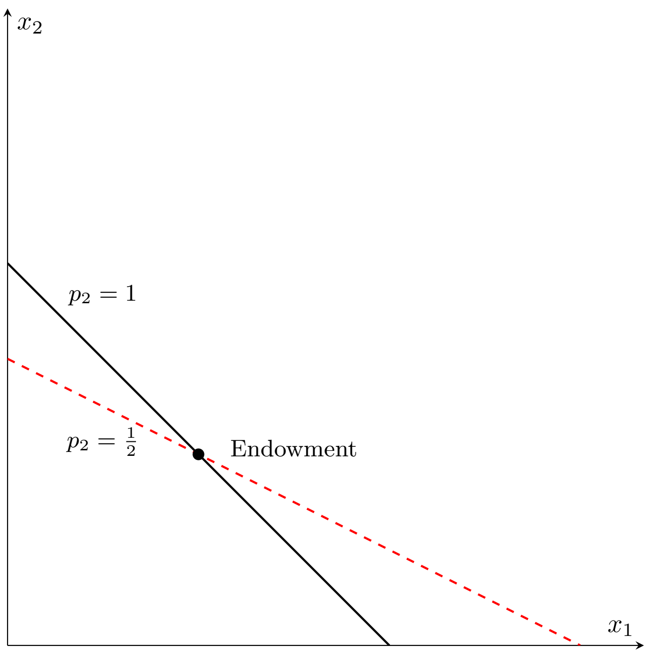


Figure 11: An example of how a change in p_2 affects the budget equation with endowment $(w_1, w_2) = (3, 3)$.

9.4 Price Changes and Net Buyers/Sellers

Unlike with a model where the consumer has an income of money, when a consumer has an endowment, there are some situations where we can say for sure how a price change affects them.

> **Info 9.1: Net Buyers/Sellers and Price Changes.**
> For a consumer who is a **net buyer** of a good, if the price of that good **decreases**, they will *remain a net buyer and will be strictly better off.*
>
> A consumer who is a **net seller** of a good, if the price of that good **increases**, they will *remain a net seller and be strictly better off.*

Look at Figure 9.3 where the price of p_2 decreases. Suppose that a consumer is a net buyer of x_2 before the price decrease. Then they are consuming to the right of the endowment point on the black line. Notice how after the price change, whatever bundle they were consuming is now on the interior of the new budget set, and new bundles that include more x_1 and x_2 become available to them. This is the core intuition for the facts above, but we will discuss the proof in more detail in class.

9.5 Example Problem

Suppose that we have an apple farmer with an endowment of $w_1 = 10$ apples and $w_2 = 0$ crusts. Their utility function is $u = min\left\{\frac{1}{2}x_1, x_2\right\}$. Initially, the prices are $p_1 = 1$, $p_2 = 1$.

The farmer's budget equation is $1x_1 + 1x_2 = 1\,(10) + 1\,(0)$ or $x_1 + x_2 = 10$. With this we can solve for demand.

The *No Waste Condition* is $\frac{1}{2}x_1 = x_2$. The budget condition is $x_1 + x_2 = 10$. Solving these two equations gives us: $x_1 = \frac{20}{3}, x_2 = \frac{10}{3}$.

At these prices, the farmer is a seller of apples x_1 and a buyer of crusts x_2. If the price of apples increases or the price of crusts decreases, the consumer will remain a seller of apples x_1 and a buyer of crusts x_2 and will be strictly better off.

9.6 Key Topics

- Set up the budget line when income is in terms of an endowment of goods rather than money.
- Know how a budget line changes when a price changes.
- Know the difference between net and gross demand and determine which is true in a problem.
- Know what it means to be a net buy or net seller and determine which is true in a problem.
- Solve demand problems involving an endowment of goods.
- Know that net buyers/sellers will remain net buyers/sellers after a price decrease/increase and be better off according to Info Box 9.1.

- Set up the budget line when income is in terms of an endowment of goods rather than money as in *Exercises 9.1, 9.2, 9.4*.
- Know how a budget line changes when a price changes as in *Exercise 9.1*.
- Know the difference between net and gross demand and calculate each as in *Exercises 9.1-9.4*.
- Know what it means to be a net buyer or net seller and determine which is true in a problem as in *Exercises 9.2- 9.4*.
- Solve demand problems involving an endowment of goods as in *Exercises 9.1-9.4*.
- Know that net buyers/sellers will remain net buyers/sellers after a price decrease/increase and be better off as in *Exercises 9.1, 9.4* and according to Info Box 9.1.

10 Intertemporal Choice

10.1 Bundles: Consumption Today and Consumption Tomorrow

We can use the endowment model from the last chapter to study borrowing and saving behavior. Consider a two-period model where consumption occurs in two periods. The consumption bundle is denoted (c_1, c_2), where c_1 represents consumption in period 1 (today) and c_2 represents consumption in period 2 (tomorrow). Similarly, the endowment of income is denoted (m_1, m_2), where m_1 is income in period 1 and m_2 is income in period 2.

> **Definition 10.1: Consumption Bundle.** The **consumption bundle** (c_1, c_2) represents the amount consumed by a consumer in period 1 (c_1) and period 2 (c_2).

> **Definition 10.2: Income Endowment.** The **income endowment** (m_1, m_2) represents the consumer's income in period 1 (m_1) and period 2 (m_2).

10.2 Budget Constraint

The "price" in this model is determined by the interest rate r, which is the rate at which the consumer can either borrow or save.

> **Definition 10.3: Interest Rate.** The **interest rate** r is the rate at which borrowing increases future obligations and saving increases future consumption.

For example, if the consumer borrows \$1000 in period 1, they must repay $1000(1 + r)$ in period 2. Similarly, if the consumer saves \$1000 in period 1, they receive $1000(1 + r)$ in period 2.

With this, we can derive the budget equation. Suppose that the consumer saves money in period 1 such that $m_1 - c_1 > 0$. Consumption in period 2 then equals their period 2 endowment m_2 plus their savings $(m_1 - c_1)$ multiplied by $1 + r$:

$$c_2 = m_2 + (1 + r)(m_1 - c_1).$$

Similarly, if the consumer borrows money such that $c_1 - m_1 > 0$, consumption in period 2 equals their endowment m_2 minus the repayment $(1 + r)(c_1 - m_1)$:

$$c_2 = m_2 - (1 + r)(c_1 - m_1).$$

Both expressions simplify to the same budget equation:

$$c_2 = m_2 + (1 + r)(m_1 - c_1).$$

We can re-write it to make it look more like the budget equations from the previous chapter.

$$(1 + r)c_1 + c_2 = (1 + r)m_1 + m_2.$$

This form measures everything in terms of period 2 consumption. The right side of this equation is **future value of income** $(1+r)m_1 + m_2$. That is, the amount of consumption the consumer can have in period 2 if they only consume in period 2.

Alternatively, we can express the budget in terms of the **present value of income** by dividing both sides by $1+r$:

$$c_1 + \frac{c_2}{1+r} = m_1 + \frac{m_2}{1+r}.$$

The right side of this is the present value of income. That is, the amount of consumption the consumer can have in period 1 if they only consume in period 1.

> **Definition 10.4: Budget Equation with Interest Rate.** The consumer's intertemporal budget equation is given by:
>
> $$(1+r)c_1 + c_2 = (1+r)m_1 + m_2.$$

10.3 Plotting the Budget Line

We can plot the budget equation $(1+r)c_1+c_2 = (1+r)m_1+m_2$ as usual. The slope of the budget line is $-(1+r)$. The c_1 intercept is the present value of income $m_1 + \frac{m_2}{1+r}$. The c_2 intercept is the future value of income $(1+r)m_1 + m_2$.

We distinguish between borrowers and savers based on whether consumption in period 1 is larger or smaller than period 1 income:

> **Definition 10.5: Borrower.** A consumer is a **borrower** if they consume more than their period 1 income: $c_1 > m_1$.

> **Definition 10.6: Saver.** A consumer is a **saver** if they consume less than their period 1 income: $c_1 < m_1$.

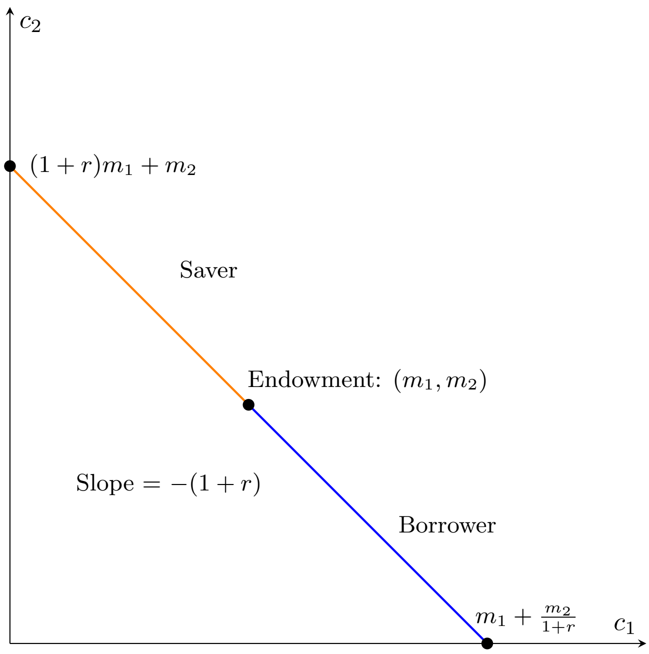


Figure 12: The budget line for a model of saving and borrowing.

10.4 Interest Rate Changes

As the interest rate goes up, the price of consuming in period 1 goes up relative to consuming in period 2.

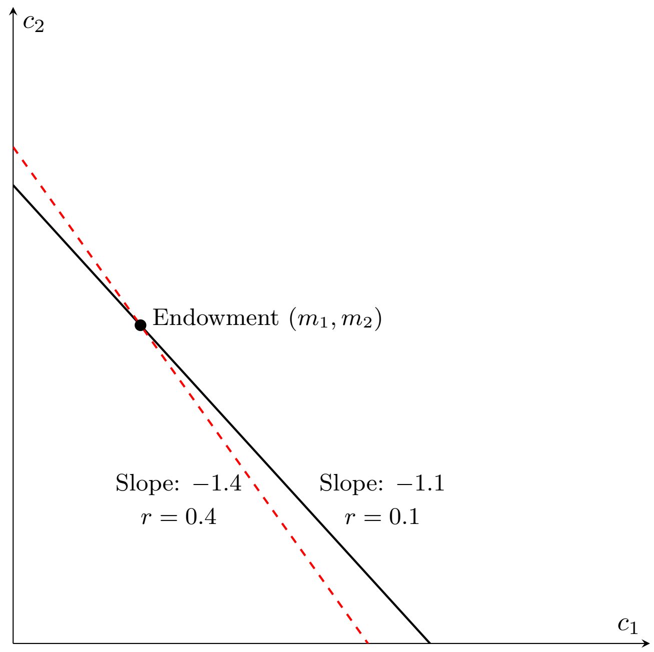


Figure 13: An example of how a change in interest rate r affects the budget equation with endowment (m_1, m_2). The black line represents the budget when $r = 0.1$. The red dotted line represents the budget after the interest rate increases to $r = 0.4$.

As in the last chapter, we can also say something quite strong about what happens to borrowers/savers when the interest rate decreases/increases.

> **Info 10.1: Borrowers and Interest Rates.**
> If a consumer is a **borrower** and the interest rate r decreases, they will **remain a borrower** and be **strictly better off**.
>
> If a consumer is a **saver** and the interest rate r increases, they will **remain a saver** and be **strictly better off**.

10.5 Example Problem

Suppose $m_1 = 200$, $m_2 = 600$, and $r = \frac{1}{2}$. The consumer's utility function is:

$$u(c_1, c_2) = c_1 c_2.$$

We start with the budget equation:

$$(1 + r)c_1 + c_2 = (1 + r)m_1 + m_2.$$

Plugging in $r = \frac{1}{2}$, $m_1 = 200$, and $m_2 = 600$:

$$\frac{3}{2}c_1 + c_2 = \frac{3}{2} \times 200 + 600 = 900.$$

Note that if the consumer only consumes in period 1 then ($c_2 = 0$):

$$c_1 = m_1 + \frac{m_2}{1+r} = 600.$$

If the consumer only consumes in period 2 then ($c_1 = 0$):

$$c_2 = (1+r)m_1 + m_2 = 900.$$

Since this consumer has a Cobb Douglass utility function which we can take partial derivatives of, to find the optimal bundle we start with the tangency condition (MRS = price ratio) which is:

$$MRS = -\frac{c_2}{c_1} = -(1+r).$$

Plugging in $1 + r = \frac{3}{2}$:

$$\frac{c_2}{c_1} = \frac{3}{2}.$$

Substitute into the budget equation and solve:

$$c_1 = 300, \quad c_2 = 450.$$

Since $c_1 = 300 > m_1 = 200$, the consumer is a borrower.

If the interest rate decreases to $r = \frac{1}{4}$, the consumer will **remain a borrower** and be **strictly better off**.

10.6 Key Topics

- Set up the budget line for intertemporal choice problems as in *Exercises 10.1- 10.3*
- Know how the budget line changes when the interest rate changes as in *Exercises 10.1-10.3*
- Know what it means to be a borrower or a saver and determine which is true in a problem as in *Exercises 10.2, 10.3*
- Solve demand problems involving intertemporal choice as in *Exercises 10.1 - 10.3*
- Know that net borrowers/savers will remain net borrowers/savers after an interest rate change and be better off according to Info Box 10.1 as in *Exercise 10.1 - 10.3*

11 Market Demand and Elasticity

In this section, we study the process of aggregating individual consumer demands to derive a market demand curve.

11.1 Adding Demand Curves

Suppose that we have multiple consumers, each with a demand for good 1 and good 2. To represent both the consumer and the good, we use a subscript for the good and an additional subscript for the consumer. Throughout, we will use numbers for the goods and letters for the consumers. The set of consumers will be called J. For instance, $J = \{a, b, c\}$ would be a set of three consumers called a, b and c respectively.

For instance, the demand of consumer a for good 1 is denoted as $x_{1,a}(p_1, p_2, m_a)$, where m_a indicates the income for consumer a. Similarly, the demand of consumer b for good 2 is $x_{2,b}(p_1, p_2, m_b)$.

Definition 11.1: Market Demand. The **market demand** for a good is the sum of individual consumer demands. Suppose the set of consumers are $\{a, b, c\}$ Formally, the market demand for good 1 is:

$$X_1(p_1, p_2, m_a, \ldots, m_z) = \sum_{j \in \{a,b,c\}} x_{1,j}(p_1, p_2, m_j).$$

Similarly, the market demand for good 2 is:

$$X_2(p_1, p_2, m_a, \ldots, m_z) = \sum_{j \in \{a,b,c\}} x_{2,j}(p_1, p_2, m_j).$$

11.2 Elasticity

Elasticity measures how demand responds to changes in price, income, or the price of other goods. It provides a unit-free measure, allowing comparison across goods.

Definition 11.2: Elasticity. Elasticity is the percentage change in quantity demanded resulting from a 1% change in another variable (price, income, etc.).

Elasticity provides an intuitive way to understand how sensitive demand is to changes in price or income. To see why elasticity takes the form it does, consider a finite change in price and quantity. If the price changes from p to $p + \Delta p$ and the demand changes from x to $x + \Delta x$, the percentage change in quantity relative to the percentage change in price is:

$$\frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}}.$$

We can view this as a fraction of fractions:

$$\frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}} = \frac{\Delta x}{x} \times \frac{p}{\Delta p}.$$

This shows the ratio of percentage changes: (percent change in quantity) / (percent change in price).

As the change becomes infinitesimally small, we move from finite differences to derivatives, giving us the formal definition of price elasticity:

$$\epsilon_{i,i} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_i}{p_i}} = \frac{\partial x_i}{\partial p_i} \times \frac{p_i}{x_i}.$$

Elasticity can be applied to any relationship between variables. In economics, we focus on price, cross-price, and income elasticity because they provide insight into consumer behavior. Each type of elasticity tells us something specific:

11.2.1 Types of Elasticities

Price elasticity of demand reflects how quantity demanded changes when the price of the good itself changes. This elasticity helps businesses determine how price changes might affect their sales and revenue.

Definition 11.3: Price Elasticity of Demand. The **price elasticity of demand** measures the percentage change in demand for a 1% increase in its own price:

$$\epsilon_{1,1} = \frac{\partial x_1}{\partial p_1} \times \frac{p_1}{x_1}.$$

This tells us how responsive consumers are to changes in the price of the good itself. If demand is highly sensitive to price changes, consumers will significantly reduce purchases when prices rise.

Cross-price elasticity captures how the demand for one good responds when the price of another good changes. This elasticity identifies relationships between goods, such as whether they are substitutes or complements.

Definition 11.4: Cross-Price Elasticity. The **cross-price elasticity** measures the percentage change in demand for a 1% increase in the price of another good:

$$\epsilon_{1,2} = \frac{\partial x_1}{\partial p_2} \times \frac{p_2}{x_1}.$$

A positive value indicates substitutes, which means that an increase in the price of one good increases the demand for the other. A negative value indicates complements, where a price increase for one good reduces the demand for the other.

Income elasticity shows how demand changes as consumer income changes. It helps identify whether a good is a necessity, a luxury, or an inferior good.

Definition 11.5: Income Elasticity. The **income elasticity** measures the percentage change in demand for a 1% increase in income:

$$\eta_1 = \frac{\partial x_1}{\partial m} \times \frac{m}{x_1}.$$

A positive value suggests a normal good, where demand increases with income. A negative value suggests an inferior good, where demand falls as income rises.

—

11.3 Classifications of Price Elasticity

Price elasticity can be classified into three categories. These classifications help us understand consumer sensitivity to price changes and how demand reacts to shifts in the market.

Unit elasticity occurs when the percentage change in quantity demanded equals the percentage change in price. This means that total revenue remains constant as price changes.

Definition 11.6: Unit Elastic. **Unit Elastic** demand occurs when the absolute value of elasticity is equal to 1:

$$|\epsilon| = 1.$$

This means that a 1% price *increase* leads to a 1% *decrease* in demand. Revenue remains unchanged as price changes.

Elastic demand occurs when consumers are highly responsive to price changes. A small increase in price leads to a significant decrease in quantity demanded.

Definition 11.7: Elastic. **Elastic** demand occurs when the absolute value of elasticity is greater than 1:

$$|\epsilon| > 1.$$

This means that demand responds sharply to price changes. For example, $\epsilon = -2$ implies that a 1% price *increase* leads to a 2% *decrease* in demand. Goods with many substitutes typically exhibit elastic demand.

Inelastic demand indicates that consumers are less responsive to price changes. Even significant price changes lead to relatively small changes in quantity demanded.

Definition 11.8: Inelastic. **Inelastic** demand occurs when the absolute value of elasticity is less than 1:

$$|\epsilon| < 1.$$

This means that demand responds weakly to price changes. For example, $\epsilon = -0.5$ implies that a 1% price *increase* leads to only a 0.5% *decrease* in demand. Necessities often have an inelastic demand.

These classifications provide insight into the market dynamics. Elastic goods face significant demand changes when prices shift, while inelastic goods maintain stable demand despite price fluctuations. Understanding elasticity helps businesses and policy makers predict consumer responses to price, income, or related goods.

11.4 Example: Finding Elasticities

Consider the demand function for good 1:

$$x_1 = \frac{m}{p_1 + p_2}.$$

We will calculate the price elasticity, cross-price elasticity, and income elasticity both in general terms and for the specific case when $p_1 = 1$, $p_2 = 1$, and $m = 100$.

Price Elasticity

To find the price elasticity of demand, we differentiate x_1 with respect to p_1:

$$\frac{\partial x_1}{\partial p_1} = -\frac{m}{(p_1 + p_2)^2}.$$

Using the elasticity formula:

$$\epsilon_{1,1} = \frac{\partial x_1}{\partial p_1} \times \frac{p_1}{x_1}.$$

Substitute $x_1 = \frac{m}{p_1+p_2}$:

$$\epsilon_{1,1} = \left(-\frac{m}{(p_1 + p_2)^2} \right) \times \frac{p_1}{\frac{m}{p_1+p_2}}.$$

Simplifying the expression:

$$\epsilon_{1,1} = -\frac{mp_1}{(p_1 + p_2)^2} \times \frac{p_1 + p_2}{m} = -\frac{p_1}{p_1 + p_2}.$$

For $p_1 = 1$, $p_2 = 1$, and $m = 100$:

$$\epsilon_{1,1} = -\frac{1}{1+1} = -0.5.$$

Interpretation: A 1% increase in the price of good 1 leads to a 0.5% decrease in its demand, indicating inelastic demand.

Cross-Price Elasticity

Differentiate x_1 with respect to p_2:

$$\frac{\partial x_1}{\partial p_2} = -\frac{m}{(p_1 + p_2)^2}.$$

Using the formula:

$$\epsilon_{1,2} = \frac{\partial x_1}{\partial p_2} \times \frac{p_2}{x_1}.$$

Substitute $x_1 = \frac{m}{p_1+p_2}$:

$$\epsilon_{1,2} = \left(-\frac{m}{(p_1+p_2)^2}\right) \times \frac{p_2}{\frac{m}{p_1+p_2}}.$$

Simplifying:

$$\epsilon_{1,2} = -\frac{p_2}{p_1+p_2}.$$

For $p_1 = 1$, $p_2 = 1$, and $m = 100$:

$$\epsilon_{1,2} = -\frac{1}{1+1} = -0.5.$$

Interpretation: A 1% increase in the price of good 2 leads to a 0.5% decrease in demand for good 1, indicating complementary goods.

Income Elasticity

Differentiate x_1 with respect to m:

$$\frac{\partial x_1}{\partial m} = \frac{1}{p_1+p_2}.$$

Using the formula:

$$\eta_1 = \frac{\partial x_1}{\partial m} \times \frac{m}{x_1}.$$

Substitute $x_1 = \frac{m}{p_1+p_2}$:

$$\eta_1 = \left(\frac{1}{p_1+p_2}\right) \times \frac{m}{\frac{m}{p_1+p_2}} = 1.$$

For $p_1 = 1$, $p_2 = 1$, and $m = 100$:

$$\eta_1 = 1.$$

Interpretation: A 1% increase in income leads to a 1% increase in demand, indicating that the good is a normal good.

11.5 Key Topics

- Understanding how individual demands aggregate into market demand and calculating market demand from individual demands, as in *Exercises 11.1, 11.2*.
- Be able to interpret elasticities, as in *Exercise 11.1 - 11.4*.
- Calculate the price, cross-price, and income elasticities from demand functions, as in *Exercises 11.1 - 11.6*.
- Classifying goods based on price elasticity into elastic, inelastic, and unit-elastic, as in *Exercise 11.2 , 11.3*.

12 Equilibrium

12.1 Market Demand/Supply

In this chapter, we focus on where prices come from by looking at the market for *one good at a time*. This is called **partial equilibrium**.

A market is made up of the demand side (buyers) and the supply side (sellers). For each side, we need to know how their quantity (bought or sold) depends on the price. Market demand $Q_d(p)$ is the total amount demanded at the price p. Market supply $Q_s(p)$ is the total amount supplied at price p.

Definition 12.1: Demand. The **demand** in a market is simply the market demand function $Q_d(p^*)$.

Definition 12.2: Supply. The **supply** in a market is simply the market supply function $Q_s(p^*)$.

We have not yet discussed how firms make decisions and how their supply function is determined. For now, however, we will take the supply functions as given and study their foundations in the next chapters.

When we "plot" a market, we tend to put price p on the vertical axis. For this reason, it is also useful to define the inverse market demand $p_d(Q)$ (at what price are Q units are demanded) and inverse market supply $p_s(Q)$ (at what price are Q units are supplied).

For example, suppose that all consumers have utility $x_1 x_2$. Each consumer demands $\frac{\frac{1}{2}m_i}{p_1}$ units of x_1. If we look at the market for x_1 then demand is $Q_d(p) = \frac{\frac{1}{2}M}{p}$ and inverse demand is $p = \frac{\frac{1}{2}M}{Q_d}$.

12.2 What is an equilibrium?

An equilibrium is defined as a price p^* such that supply is equal to demand.

Definition 12.3: Equilibrium Price. The **equilibrium price** of a market is a price p^* such that demand is equal to supply $Q_d(p^*) = Q_s(p^*)$.

Definition 12.4: Equilibrium Quantity. The **equilibrium quantity** Q^* in a market is the quantity demanded and supplied at the equilibrium price. That is, $Q^* = Q_S(p^*) = Q_d(p^*)$.

We focus on situations where supply equals demand for the following reason...

Suppose that at some price p, **supply exceeded demand** $Q_s(p) > Q_d(p)$.

In this case, the price is too high. There are surplus units of the good, and any firm with a surplus unit would be willing to sell at a lower price, since otherwise it will be wasted. This created downward pressure on prices.

Suppose now that for some price p, **demand exceeds supply** $Q_d(p) > Q_s(p)$.

In this case, the price is too low. There is a shortage, and consumers are willing to buy at a higher price. There is upward pressure on prices. Thus, the only time there is no pressure for the market price to change is when supply at the price is equal to demand at the price.

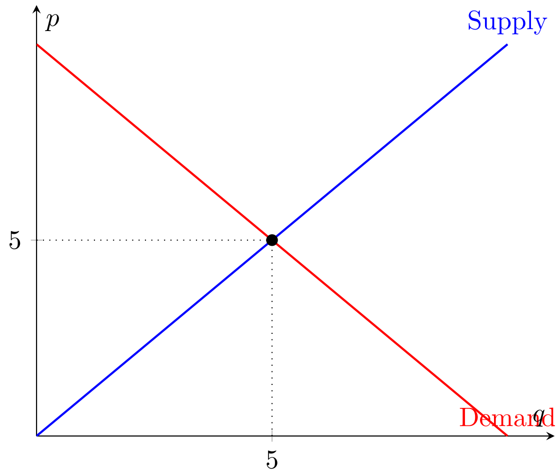


Figure 12.1: Graphical Demonstration of Equilibrium when $Q_d(p) = 10 - p$ and $Q_s(p) = p$

12.3 Examples

Let's suppose $Q_d(p) = \frac{2500}{p}$ and $Q_s(p) = 100p$.

To find the equilibrium, we look for a price p^* such that $Q_d(p) = Q_s$. This is done by solving $\frac{2500}{p} = 100p$ which has a solution $p = 5$. To get equilibrium quantity, plug this price into either supply or demand. We should get the same thing: $Q^* = Q_s(p) = Q_d(p) = 500$. Notice that we use Q^* to refer to the equilibrium quantity.

12.3.1 Fixed Supply

With **fixed supply**, the quantity supplied Q_s is constant for any price. The inverse supply curve is a vertical line. This would be the case, for instance, with concert tickets. The size of the venue is fixed regardless of the price of tickets. For example, suppose that supply is fixed at $Q_s(p) = 1000$ and demand is $Q_d(p) = \frac{500}{p}$. To find the equilibrium price, solve $1000 = \frac{500}{p}$. This gives us the equilibrium price of $p^* = \frac{1}{2}$. This is the price at which the consumer will demand the total supply of 1000. Equilibrium quantity, of course, is $Q^* = 1000$

12.4 Effect of a Tax

Suppose the government imposes a tax of t per unit of the good. If we think of p as being the price that firms charge for the good (the "sticker price") then firms will receive p for every good sold and consumer will have to pay $p+t$. This leaves us with the following equilibrium condition with a tax: $Q_s(p) = Q_d(p+t)$.

For example, suppose $Q_s(p) = 100p$ and $Q_d(p) = 300 - 50p$. The government imposes a tax of $t = 3$.

The equilibrium price without a tax is the solution to $100p = 300 - 50p$. This gives us $p^* = 2$ and $Q^* = 200$. With the tax of $t = 3$, the new equilibrium condition is $300 - 50(p+3) = 100p$. The new equilibrium price is $p^* = 1$ and new equilibrium quantity is $Q^* = 100$. Suppliers get $p = 1$ per unit and consumers pay $1 + 3 = 4$. This example is shown graphically in the figure below.

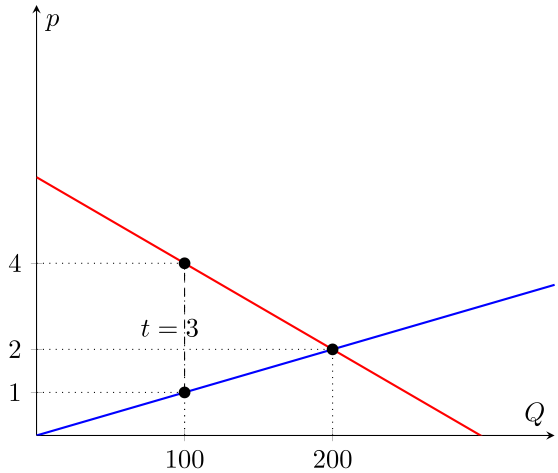


Figure 12.2: Effect of a Tax

12.4.1 Tax Burden

In this example, after the tax is imposed, consumers pay 4. Before the tax, they only paid 2. Similarly, before the tax producers got 2 per unit sold but now only get 1. Consumers pay 2 more than before and producers get 1 less than before. These differences are called the tax burden or tax incidence. They allow us to determine who ends up "paying" for the tax.

> **Definition 12.5: Tax Burden.**
> The **tax burden on consumers** is the difference between amount consumers pay after the tax is imposed (new equilibrium price plus tax) and the pre-tax equilibrium price.
> The **tax burden on producers** is the difference between pre-tax equilibrium price and the equilibrium price after the tax is imposed.

Notice that these amounts sum to the size of the tax (3 in this case). This will always be the case.

Info 12.1: Total Tax Burden. The sum of the tax burden on consumers and the tax burden on producers is the total tax t.

This also allows us to calculate the tax burden as a proportion of the tax. Just divide the burden on each "side" of the market by the size of the tax. Here, the proportion of the tax paid by consumers if $\frac{2}{3} \approx 66.67\%$ and the proportion paid by producers is $\frac{1}{3} \approx 33.34\%$.

Definition 12.6: Elasticity of Demand / Supply.

- **Elasticity of Demand** is, roughly, the percentage change in demand for a 1% change in price. It is $\frac{\partial Q_d(p)}{\partial p} \frac{p}{Q_d(p)}$
- **Elasticity of Supply** is, roughly, the percentage change in supply for a 1% change in price. It is $\frac{\partial Q_s(p)}{\partial p} \frac{p}{Q_s(p)}$

The burden of the tax is determined by the relative elasticities of supply and demand. If demand is relatively elastic and supply is relatively inelastic, then most of the burden will be on producers. This is because suppliers cannot "pass on" much of the tax to consumers. If they did, because demand is relatively elastic, demand would decrease too much. On the other hand, when demand is relatively inelastic compared to supply, most of the burden of the tax will be on the consumers. The supplier's "pass on" most of the tax to consumers because demand is inelastic. We will discuss the graphical intuition for these claims more in class.

Info 12.2: Tax Burden and Elasticity. The less elastic demand is relative to supply, the more the burden of a tax will be on consumers. The less elastic supply is relative to demand, the more the burden of a tax will be on producers.

The effect of the tax is a lower quantity, consumers pay more than they used, and suppliers receive less than they used to. Both consumers and producers are worse off under this tax. To quantify how much "worse off" we use the concept of surplus.

12.5 Surplus and Deadweight Loss

Consumer surplus is a measure of welfare that tells us how much "better-off" the consumers are because the market sells the quantity q at price p compared to if the market did not exist at all.

Definition 12.7: Consumer Surplus. The **consumer surplus** is measured by the area under the inverse demand curve but above the price they pay.

Definition 12.8: Producer Surplus. The **producer surplus** is measured by the area above the inverse supply curve but below the price they receive.

Using the areas under inverse demand but above price to measure consumer surplus is motivated by thinking of the height of the inverse demand at some point as the price *some consumer* is willing to pay for a unit of that good. The difference between that height and the price the consumer actually has to pay is a measure of how happy they are to pay less than they were willing to. This difference is thus the measurement of some consumer's surplus from buying the good at price p. "Summing" over all consumers gives that area below the inverse demand curve and above the price. The same argument motivates the area above the inverse demand and below the price as being the producer surplus.

In the tax example above, the consumer surplus (with no tax) is $\frac{1}{2}(4 * 200) = 400$. The producer surplus (with no tax) is $\frac{1}{2}(2 * 200) = 200$. To find these plot the inverse demand and supply, along with the price, and calculate the area of the resulting triangles. The total welfare is the sum of consumer and producer surplus. In this case, that is 600. These surplus calculations are shown graphically in the figure below.

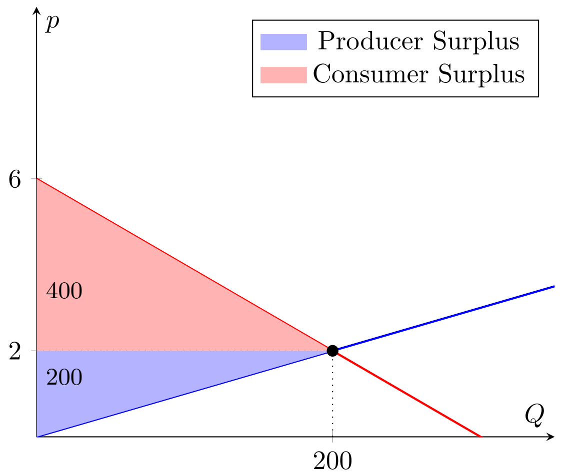


Figure 12.3: Surplus in Tax Example before Tax is Imposed.

This total surplus of 600 is actually the most we could possibly get in this market. That is because, to produce any more surplus, we would need to sell more units. However, there is no consumer left who is willing to buy at a price that a firm is willing to sell. That is because, to the left of the equilibrium, the inverse demand curve is below the inverse supply curve. Since total surplus is maximized here, there is no way to make any consumer or firm better off without making some other consumer or firm worse off. When this is the case, we say that the market has reached Pareto equilibrium. In the absence of taxes or other complicating factors, a market equilibrium will always be paretro efficient.

A tax, however, will lower the total surplus. In the example above, after the tax is imposed, the consumer surplus is $\frac{(6-4)100}{2} = 100$ and the producer surplus is: $\frac{(1)100}{2} = 50$. When there is a tax, we include the government revenue in the calculation of total surplus. This is because that revenue is not lost. It could be transferred back to the consumer or producers in some way. So, it contributes to the total surplus. The government revenue under the tax in the example above is $3 * 100 = 300$. Total surplus is $100 + 50 + 300 = 450$. Compare this to the original surplus which was 600. The difference is 150. We call this amount the dead-weight loss. It measures the amount of total surplus lost due to a tax. This dead-weight loss occurs because the tax prevents some firms and consumers from trading even though there is some price at which they would both be happy to trade at.

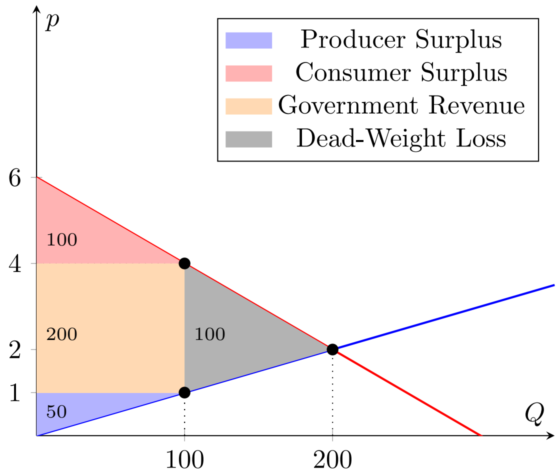


Figure 12.4: Surplus in Tax Example after Tax is Imposed.

12.6 Key Topics

- Understand the definition of market equilibrium as the price at which market demand equals market supply ($Q_d(p) = Q_s(p)$).
- Find equilibrium prices and quantities analytically, as in *Exercises 12.1 - 12.4*
- Analyze the impact of a per-unit tax on market equilibrium, including the wedge between consumer price and producer price, as in *Exercises 12.1 - 12.4*
- Calculate and interpret consumer surplus, producer surplus, and dead-weight loss, and understand how these welfare measures change under taxation, as in *Exercises 12.1 - 12.4*
- Demonstrate equilibrium problems graphically, including the effect of a tax and the areas of surplus, government revenue, and dead-weight loss, as in *Exercises 12.1 - 12.4*

13 Production

We now begin our study of the supply side of the market. Our first task is to model firms in a mathematical and abstract way. The nature of a firm is that they use inputs to produce outputs, and then they sell those outputs to consumers in order to maximize their profits. At least that is how we will think of them in this class. In this chapter, we focus only on defining the process by which firms turn inputs into outputs. We do this by defining a production function.

A production function describes how much output y are produced from each possible **input bundle** (x_1, x_2).

Definition 13.1: Input Bundle. An **input bundle** (x_1, x_2) describes how much of input 1 (x_1) and input 2 (x_2) are used for production.

Definition 13.2: Production Function. A **production function** maps an amount of each input bundle into an amount of output. Generically, we will write it like this: $f(x_1, x_2)$.

For example, a baker can always take 2 apples and 1 crust and make a pie. Suppose x_1 represents crusts and x_2 represents apples. For instance, input bundle $(2, 1)$ produces 1 pie so we write: $f(2, 1) = 1$ Similarly, $f(4, 2) = 2, f(6, 3) = 3$ and so on. Generically, we can write: $f(x_1, x_2) = min\left\{\frac{1}{2}x_1, x_2\right\}$. This function will tell us exactly how many pies any input bundle will produce.

Note that while production functions are very similarly to utility functions, which also turn bundles into a number. The production functions are **cardinal**. The numbers are meaningful. For consumers, we could do something like multiply a utility function by two and the result would still represent the same preferences. But if we multiply a production function by two, that does not represent the same technology, that represents a technology that is two times more efficient!

Most of the utility functions we have looked at so far are also common production functions. For example, $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ is an instance of a Cobb Douglass production function. $f(x_1, x_2) = min\{x_1, x_2\}$ is an instance of a perfect compliments production function. $f(x_1, x_2) = x_1 + x_2$ is an instance of perfect substitutes production.

13.1 Short-Run

Sometimes, a firm cannot change one of its inputs. When this is the case, we refer to the situation as **short-run**.

Definition 13.3: Short-Run. We refer to any situation in which a firm cannot change one of its inputs as being in **short-run**.

For instance, suppose a firm has production function $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ and x_2 is fixed at $\bar{x}_2 = 4$ in the short-run. Then the firm's short-run production function is: $f(x_1, \bar{x_2}) = f(x_1, 4) = 2x_1^{\frac{1}{2}}$.

13.2 Isoquants

Isoquants are combinations of input that give you the same amount of output. They are analogous to indifference curves for consumers.

> **Definition 13.4: Isoquant.** An **isoquant** is a set of input bundles that all produce the same output.

Think of them as recipes for the same output. Let's look at the baker example again. Two apples and one crust $(2,1)$ makes one pie, but so does $(3,1),(4,1),(2,2),(2,3)$. These are all input bundles on the isoquant for 1 unit of output. Isoquants for this production function are plotted below for $y = 1,2,3,4$.

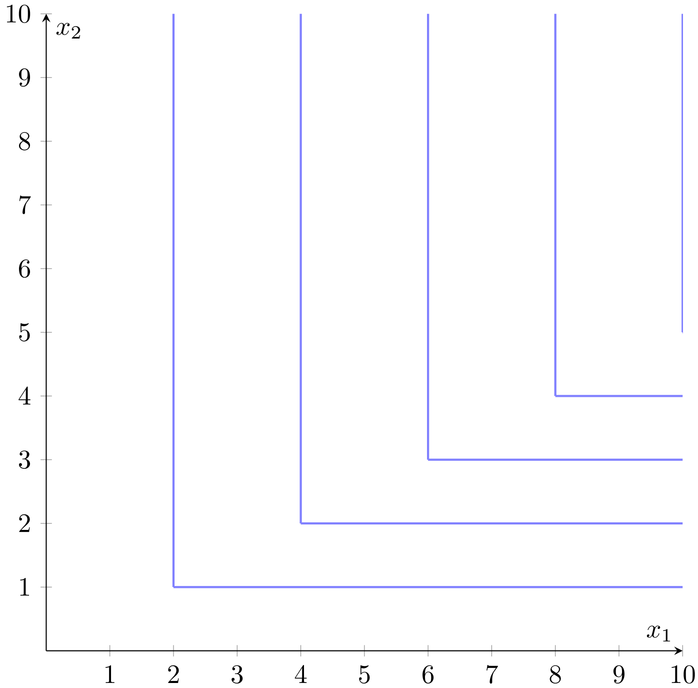


Figure 13.1: Isoquants for $min\left\{\frac{1}{2}x_1, x_2\right\} = y$ **for** $y = 1,2,3,4$

Working with isoquants is identical to working with indifference curves. The techniques we learned for finding isoquants, sketching them, analyzing them, etc. are the same, so we will not dwell on them too much here.

13.3 Marginal Products

When we talked about utility functions, the partial derivatives (the marginal utilities) were useful in finding the marginal rate of substitution, but were not that useful on their own. This is because "how much" utility increases when we increase a good is not meaningful since "how much" utility is not itself meaningful. The marginal utilities are only meaningful in comparison to each other. However, the amount of production is meaningful and tangible. 5 pies is 5 pies while 5 points of utility is not tangible.

Because of the fact that the amount of production is a meaningful number, how that number changes when we change one of the inputs is meaningful information. These are the marginal products. They are measured by the partial derivatives of the production function.

> **Definition 13.5: Marginal Product.** The **marginal product** of input i is $MP_i = \frac{\partial f(x_1,x_2)}{\partial x_i}$. This measures how production increases when x_i is increased by a small amount. Roughly speaking, how much extra output you get by increasing input x_i by one unit holding other inputs fixed.

For example, suppose $f(x_1, x_2) = 2x_1 + x_2$. $MP_1 = 2, MP_2 = 1$. Roughly speaking, if we increase input 1 by one unit (holding x_2 fixed), we get 2 more units of output. If we increase x_2 by one unit (holding x_1 fixed) we get 1 more unit of output.

As another example, suppose $f(x_1, x_2) = (x_1 + x_2)^{\frac{1}{2}}$. The name of this function is the CES production function (Constant Elasticity of Substitution). Don't worry about what constant elasticity of substitution means just yet, we will discuss it later if there is time. The marginal products are $MP_1 = \frac{\partial\left((x_1+x_2)^{\frac{1}{2}}\right)}{\partial x_1} = \frac{1}{2}\frac{1}{\sqrt{x_1+x_2}}$ and $MP_2 = \frac{1}{2}\frac{1}{\sqrt{x_1+x_2}}$. Notice that the extra output for increasing either of the inputs only depends on the sum of the input amounts $x_1 + x_2$ and is decreasing in both.

As a final example, consider $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ (Cobb Douglass Production). $MP_1 = \frac{\partial\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)}{\partial x_1} = \frac{1}{2}x_1^{\frac{1}{2}-1}x_2^{\frac{1}{2}} = \frac{1}{2}x_1^{-\frac{1}{2}}x_2^{\frac{1}{2}} = \frac{1}{2}\frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} = \frac{\sqrt{x_2}}{2\sqrt{x_1}}$ and $MP_2 = \frac{\sqrt{x_1}}{2\sqrt{x_2}}$. The marginal product of 1 is decreasing in x_1 but increasing in x_2 and vice versa.

13.4 Diminishing Marginal Product

Diminishing marginal product is the property that implies that each input becomes less productive the more you use (holding other inputs constant).

> **Definition 13.6: Diminishing Marginal Product.** A production function has **diminishing marginal product** for input i if MP_i is decreasing in x_i.

For the Cobb Douglass production function $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, $MP_1 = \frac{\sqrt{x_2}}{2\sqrt{x_1}}$. Since x_1 only appears in the denominator of this, MP_1 is decreasing in x_1 and thus, this production function has diminishing marginal product for x_1. The same goes for x_2. However, not all production functions have diminishing marginal product. For instance, the Cobb Douglass production function $x_1^2 x_2^2$ has *increasing* marginal product for both x_1 and x_2 since $MP_1 = 2x_1x_2^2$ which is increasing in x_1. The same goes for x_2.

13.5 Returns to Scale

While marginal product measures how production changes as we change one of the inputs, the returns to scale measures how production changes when all of the inputs are scaled up. Take our baker example. If we start with the input bundle of 2 apples, 1 crust $(2, 1)$ we get 1 pie. If we double either of the inputs, we still get one pie. For instance, $f(2, 2)$ and $f(4, 1)$ both give 1 pie. However, *if we double both inputs* to $(4, 2)$, we get 2 pies. Doubling the inputs doubles the outputs. We call this linear returns to scale. However, for some production functions, when we double the inputs, we get less than double the outputs (decreasing returns to scale) or when we double inputs, we get more than double the output (increasing returns to scale).

Definition 13.7: Returns to Scale. For any $t > 1$:

Liner (constant) returns to scale requires: $f(tx_1, tx_2) = tf(x_1, x_1)$.

Decreasing returns to scale requires: $f(tx_1, tx_2) < tf(x_1, x_1)$.

Increasing returns to scale requires: $f(tx_1, tx_2) > tf(x_1, x_1)$.

Consider $f(x_1, x_2) = x_1 x_2$. Let's start with $(1, 1)$. This produces $f(1, 1) = 1$ unit of output. If we double both inputs to $(2, 2)$, then production is $f(2, 2) = 4$. Notice that output more than doubles! To see that this is true in general for any bundle and any t:

$$f(tx_1, tx_2) = tx_1 tx_2 = t^2 x_1 x_2 = t^2 f(x_1, x_2)$$

Since $t^2 > t$ for any $t > 1$, whatever bundle we start with and whatever t¿1 we choose, scaling the input by t will scale the output by more than t!

13.6 Technical Rate of Substitution

Along a particular isoquant, the slope measures how much x_2 you can give up if you add 1 unit of x_1 so that you continue producing the same amount of output. This slope and trade-off are measured by the **Technical Rate of Substitution** or **TRS**. The TRS for producers is analogous to the marginal rate of substitution, MRS for consumers.

Definition 13.8: Technical Rate of Substitution.
The **Technical Rate of Substitution**, **TRS** for production function $f(x_1, x_2)$ is the negative of the ratio of marginal products. $TRS = -\frac{\frac{\partial f(x_1, x_2)}{\partial x_1}}{\frac{\partial f(x_1, x_2)}{\partial x_2}} = -\frac{MP_1}{MP_2}$.

The TRS measures the trade-offs that a firm is *willing* to make. Eventually, it will play a key role in finding input bundles that minimize costs.

13.7 Key Topics

- Understand the role of the production function in describing how input bundles turn into output.

- Understand the difference between short-run and long-run production; recognize that in the short-run at least one input is fixed.
- Understand isoquants, which represent all combinations of inputs that yield the same output as in *Exercise 13.1.*
- Calculate marginal products by taking partial derivatives of the production function, and interpret marginal product as the increase in production for a small increase in one input as in *Exercise 13.2, 13.5.*
- Understand the concept of diminishing marginal product and identify when an input has diminishing marginal product as in *Exercise 13.3.*
- Understand and interpret returns to scale, which measures how output changes when all inputs are scaled by a constant factor (as in *Exercise 13.4*)
- Distinguish between constant, increasing, and decreasing returns to scale as in *Exercise 13.4.*
- Compute and interpret the Technical Rate of Substitution (TRS), the slope of isoquants, as the rate at which a firm can reduce the use of input x_2 if they increase the input x_1 while maintaining the same level of output as in *Exercise 13.5.*

14 Cost Minimization

14.1 Motivation- Profit Maximization

In this class, we will assume that a firm's goal is to maximize profits. In reality, firms may have a more complex objective, but profit is certainly a large part of that objective for almost all firms, and we will gain substantial insight by abstracting firm behavior down to pure profit-maximization.

The profit function of a firm is made up of revenue and costs. Revenue is the price the firm receives multiplied by the output of the firm. Usually, the price a firm receives is a function of how much it produces. How this price is determined depends on the details of the market. We will look at some possibilities in later chapters. But for now, we will just leave this as some unknown price function $p(y)$. Output is simply y.

Thus, **firm revenue** in terms of y is given by: $p(y)\,y$.

Notice that since $y = f(x_1, x_2)$, we could also write revenue in terms of the amounts of inputs used instead of outputs.

Thus, **firm revenue** in terms of inputs (x_1, x_2) is given by: $p\Big(f(x_1, x_2)\Big) f(x_1, x_2)$.

Costs are determined by the amount of inputs used and the price of those inputs. Here, we will assume the price of inputs is fixed at w_1 and w_2. (We use w because price of inputs are often called "wages").

> **Definition 14.1: Input Prices.** The price of x_1 per unit is w_1 and the price of x_2 per unit is w_2.

Thus, the **cost of input bundle** (x_1, x_2) is $w_1x_1 + w_2x_2$. This leads to the following definition of the profit function.

> **Definition 14.2: Profit Function in Terms of Input Bundle.**
> A firm's **profit** in terms of input bundle (x_1, x_2) is
>
> $$\pi(x_1, x_2) = p(f(x_1, x_2))\, f(x_1, x_2) - (w_1x_1 + w_2x_2)$$

14.2 Profit Maximization Requires Cost Minimization

In the long-run, a firm's goal is to maximize their profit by choosing x_1 and x_2. However, we can simplify the problem of maximizing profit by breaking it down into two steps and using the following observation.

> **Definition 14.3: Cost Minimizing Bundle.** The bundle (x_1, x_2) that **minimizes the cost** of producing output y is the bundle that minimizes $w_1x_1 + w_2x_2$ subject to the constraint that $f(x_1, x_2) = y$.

Info 14.1: Profit Max / Cost Min. Profit maximization implies cost minimization.

To see this, notice whatever x_1 and x_2 maximize profit, there is some amount of output produced y^* this is the profit maximizing level of output. To produce this amount of output, the firm must use some bundle inputs on the isoquant $f(x_1, x_2) = y^*$. However, if they choose any input bundle that is not the cheapest one, then it could produce the same output and get the same revenue while reducing cost simply by instead using the cheapest bundle that produces y^*. This would lead to an increase in profit. Thus, if a firm was not minimizing cost of producing what they thought was the profit maximizing level of output, there is a cheaper way to earn the same revenue, and thus get more profit.

This lets us break down the profit maximization problem into two steps:

Info 14.2: Two-Step Profit Maximization.

1. Calculate the cheapest way to produce any level of output y.
2. Calculate the most profitable y.

Step 1 is what we will focus on in this chapter: finding the **cost minimizing** bundle.

Notice, when we minimize cost to complete this step, we can ignore revenue, which depends on that pesky function $p(y)$ and allows us to put off talking about how price depends on output.

14.3 Cost Minimization

When we discussed utility maximization, we argued that a bundle that maximizes utility must be on an indifference curve that does not cross through the budget line. A very similar property will hold for cost minimization.

Definition 14.4: Isocost Line. An **isocost** line is a set of input bundles (x_1, x_2) that cost the same. They are defined by equations like $w_1x_1 + w_2x_2 = c$. This is the equation for the isocost line of bundles that cost c.

Definition 14.5: Isoquant. An **isoquant** is a set of input bundles (x_1, x_2) that produce the same amount of output. They are defined by equations like $f(x_1, x_2) = y$. This is the equation for the isoquant that produce output y.

Isocost lines are somewhat analogous to budget lines. They are straight lines with slope $-\frac{w_1}{w_2}$. An isoquant is somewhat analogous to an indifference curve.

Recall that utility maximization essentially tried to find the bundle on the budget line that is also on the highest indifference curve (most utility). What a firm does in trying to find a cost minimizing bundle for producing output y is to instead look for a bundle of inputs that is on the isoquant for output y but is on the **lowest isocost**.

Info 14.3: Cost Minimizing Isocost Does not Cross Isoquant. The isoquant cannot pass below the isocost curve of the cost minimizing bundle.

This fact is analogous to the similar fact for utility maximization that the indifference curve through the utility maximizing bundle cannot pass below the budget line (See Info Box 6.1).

Let's investigate this claim using the figure below. Notice that the bundle x is on the isoquant for y units of output. That is, input bundle x produces output y as required. So, it produces the right amount of output, but notice the isoquant passes below the isocost of x. x' is another input bundle that is on the isoquant. Thus is also produces the same output, but it costs **strictly less**! Thus, x could not possibly be cost minimizing input bundle for producing y. In this case, x' is the cost minimizing bundle.

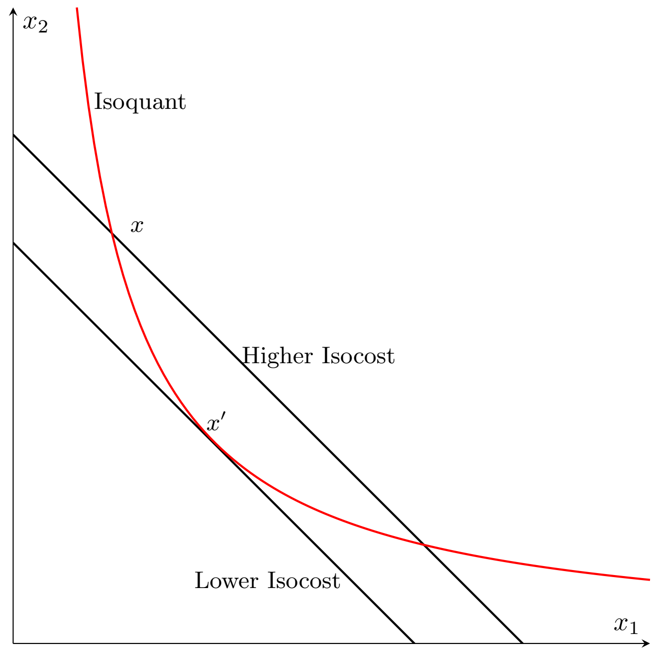


Figure 14: Demonstration of cost minimization. The bundle x' is cost minimizing.

The fact that a cost minimizing bundle must lie on an isocost that does not cross through the isoquant for y has some important implications for finding the optimal input bundle.

Info 14.4: Three Possibilities for a Cost Minimizing Input Bundle. When the production function is monotonic, the cost minimizing bundle must meet one of the following three conditions.

1. (**Tangent**) It is at a point where the isocost curve has the same slope as the isoquant at that point.
2. (**Touching but not Smooth**) The point is a "non-smooth" point on the isoquant, but that point just touches the isocost.
3. (**Boundary**) The point is at one of the boundaries where either $x_1 = 0$ or $x_2 = 0$.

As long as the production function is smooth (we can take its derivatives), we are in the territory of possibility 1 above. The cost minimizing bundle *must occur* where the slope of the isoquant is equal to the slope of the isocost. That is, the **tangency condition** must be met. Let's look more at these slopes.

Definition 14.6: Technical Rate of Substitution. The **technical rate of substitution** (**TRS**) is the slope of a firm's isoquant at a particular point. It measures, roughly, how much the firm can reduce input x_2 to maintain the same level of production if they increase x_1 by one unit.

The TRS is measured by the ratio of marginal products $\frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}}$.

Definition 14.7: Slope of Isocost. The slope of an isocost curve measures, roughly, how much a firm has to reduce x_2 by to maintain the same cost by if they increase x_1 by one unit.

Thus, the tangency condition is $TRS = -\frac{w_1}{w_2}$.

14.3.1 Interpreting the Tangency Condition

$TRS = -\frac{w_1}{w_2}$. is identical to $-\frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}} = -\frac{w_1}{w_2}$ which can also be written $\frac{MP_1}{w_1} = \frac{MP_2}{w_2}$. This equation implies that the cost of increasing output by one unit using x_1 $\left(\frac{MP_1}{w_1}\right)$ is the same as the cost of increasing output by one unit using x_2 $\left(\frac{MP_2}{w_2}\right)$. This of these as "productivity per dollar". If the productivity per dollar of all the inputs is not the same, then you can decrease use of the less productive input, increase the use of the more productive input to maintain the same level of output while saving cost.

I hope you will find this to be an intuitive condition for cost minimization. If it were not the case, the firm could reduce their use of the more expensive input (per unit of output), and increase their use of the less expensive (per unit of output) input and lower their cost while producing the same amount.

14.4 Minimizing Cost for a Cobb Douglass Production Function

Since the mathematical conditions for cost minimization are so similar to the conditions for maximizing utility, you will find the examples to be very familiar. For instance, let's minimize the cost of producing y units of output with production function $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$ when $w_1 = 1$ and $w_2 = 1$.

The TRS is $-\frac{\frac{\partial\left(x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}\right)}{\partial x_1}}{\frac{\partial\left(x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}\right)}{\partial x_2}} = \frac{\frac{1}{4} x_1^{-\frac{3}{4}} x_2^{\frac{1}{4}}}{\frac{1}{4} x_1^{\frac{1}{4}} x_2^{-\frac{3}{4}}} = -\frac{x_2}{x_1}$. This gives us the tangency condition $-\frac{x_2}{x_1} = -\frac{w_1}{w_2}$. Since $w_1 = 1$ and $w_2 = 1$ this simplifies to $x_1 = x_2$.

Instead of plugging this into a budget equation like we would for the consumer utility maximization, we need to plug it into the producer's constraint, the production constraint, which is $x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} = y$. Plug in the condition above for x_2, we get: $x_1^{\frac{1}{4}} x_1^{\frac{1}{4}} = y$. Solving this for x_1 gives us $x_1 = y^2$. This is the so-called **conditional factor demand** for x_1. This is the amount of x_1 to use to produce y in the cheapest way possible. Plugging this back into the tangency condition, the conditional factor demand for x_2 is $x_2 = y$.

Definition 14.8: Conditional Factor Demand.
The **conditional factor demand** for an input $x_i^*(y, w_1, w_2)$, is the amount of x_i that is used to produce output y in the cheapest way possible when prices are w_1, w_2.

To calculate the cheapest cost for producing y, calculate the cost of the conditional factor demands using $w_1 x_1 + w_2 x_2$. Plugging these demands in gives us: $y^2 + y^2 = 2y^2$. This is the so-called **cost function** for the producer. $c(y) = 2y^2$. This is the amount it costs to produce y in the cheapest way possible. It is a very important function.

Definition 14.9: Cost Function.
The **cost function** $c(y)$ is the total cost of producing output y in the cheapest way possible. It is calculated as $c(y) = w_1 x_1^* + w_2 x_2^*$ where x_1^* and x_2^* are the conditional factor demands for input 1 and 2.

14.4.1 Marginal Cost

Notice in the previous example, $c(1) = 2$, $c(2) = 8$, $c(3) = 18$. Each additional unit of output is getting more and more expensive. We can formalize this by looking at the derivative of the cost function with respect to y which measures how cost grows as we increase y. In this case it is $\frac{\partial c(y)}{\partial y} = 4y$. This is known as the **marginal cost**.

Definition 14.10: Marginal Cost. **Marginal Cost**, $mc(y) = \frac{\partial c(y)}{\partial y}$ measures, approximately, how much extra the next unit of output will cost when the first is currently producing y units.

In this case, we have **increasing** marginal cost since $mc(y)$ is increasing in y. This is perhaps unsurprising because the production function that generates this $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$ has diminishing marginal product. Doubling input will less than double output. Thus, to double output, you need to more than double input. In this case, that means continuing to increase the output will become more and more expensive.

Definition 14.11: Categories of Marginal Cost.

- **Constant Marginal Cost** $mc(y)$ is constant in y
- **Increasing Marginal Cost** $mc(y)$ is increasing in y
- **Decreasing Marginal Cost** $mc(y)$ is decreasing in y

14.5 Minimizing Cost for a Perfect Complements Production Function

Consider a perfect complements production function $f(x_1, x_2) = \min\{3x_1, 2x_2\}$ where $w_1 = 1$ and $w_2 = 1$.

To produce y units of output, the firm must satisfy the production constraint

$$\min\{3x_1, 2x_2\} = y.$$

Since this is a perfect complements production function, the minimum cost input bundle must meet the **no-waste condition**.

$$3x_1 = 2x_2$$

This condition reflects the fact that inputs must be used in fixed proportions. Any deviation from these proportions would not increase output but would only incur additional cost, making the chosen combination the cost-minimizing one.

Plugging this condition into the produciton constraint gives us $min\{3x_1, 3x_2\} = y$ or $3x_1 = y$. Solving this for x_1 gives us $x_1 = \frac{y}{3}$. Plugging this back into the no-waste condition, we get $x_2 = \frac{y}{2}$. Thus the conditional factor demands are:

$$x_1^* = \frac{y}{3} x_2^* = \frac{y}{2}$$

Substituting these optimal inputs into the cost function gives the minimum cost of producing y units:

$$c(y) = w_1 \frac{y}{3} + w_2 \frac{y}{2}$$

Since $w_1 = 1$ and $w_2 = 1$

$$c(y) = \frac{y}{3} + \frac{y}{2} = \frac{5}{6} y$$

Here, we have constant marginal costs $mc(y) = \frac{5}{6}$. Every additional unit of output costs exactly $\frac{5}{6}$ extra dollars.

14.6 Minimizing Cost for a Perfect Substitutes Production Function

Now, consider a production function with perfect substitutes given by $f(x_1, x_2) = 2x_1 + x_2$ where $w_1 = 1$ and $w_2 = 1$.

The production constraint to produce y units of output is

$$2x_1 + x_2 = y.$$

Because the inputs are perfect substitutes, a cost minimizing bundle will be either using all x_1 or all x_2. We just need to figure out whether using all x_1 or all x_2 to produce y is cheaper.

Suppose they use only x_1 ($x_2 = 0$). Then their production is $f(x_1, 0) = 2x_1$. To meet the production constraint $2x_1 = y$, they need to use $x_1 = \frac{y}{2}$ The cost of this is $w_1 \frac{y}{2}$ which here is $\frac{y}{2}$. On the other hand if they use only x_2 their production constraint is $x_2 = y$ and the cost of that is y. Notice it is cheaper to use only x_1 and so that is what they will do.

Their conditional factor demands are:

$$x_1^* = \frac{y}{2}, x_2 = 0$$

Calculating the cost of this optimal bundle gives us $c(y) = \frac{y}{2}$. Again, we have constant marginal costs $mc(y) = \frac{1}{2}$.

14.7 Short-Run Costs

In the example where we had $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$ and where $w_1 = 1, w_2 = 1$ the cost function was $c(y) = 2y^2$. However, suppose x_2 is something like the number of machines a company uses and that it is difficult to quickly change the number of machines in a factory. In the **short-run** the number of machines might be fixed.

Definition 14.12: Short-Run. A use the term **short-run** anytime an input is fixed and not freely adjustable by the firm.

Definition 14.13: Long-Run. A use the term **long-run** when all inputs are freely adjustable by the firm.

Suppose in the short-run x_2 is fixed at $x_2 = 16$. Then in the short run, the firm's production function is $f(x_1, 4) = 2x_1^{\frac{1}{4}}$. How much does it cost to product y in the short run?

To produce y the firm needs to determine how much x_1 to use. Here there is no cost-minimization step. There is only one level of x_1 that will produce y when x_2 is fixed. It is the solution to $2x_1^{\frac{1}{4}} = y$ or $x_1 = \frac{1}{16}y^4$.

Since the company incurs the cost of this $x_1 = \frac{1}{16}y^4$ and their fixed amount of input two, $x_2 = 16$ their **short-run** cost is $c(y) = w_1 \frac{1}{16}y^4 + 16w_2$. Assuming $w_1 = 1$ and $w_2 = 1$, we have the short-run cost $c(y) = \frac{1}{16}y^4 + 16$.

Compare this to the long-run cost $c(y) = 2y^2$. Look at the table below. Notice how the short-run cost is greater than the long-run cost for all but $y = 4$. At $y = 4$, the conditional factor demand for $x_2^* = 16$. So if the firm wants to product $y = 4$, then having a fixed $x_2 = 16$ is not a problem. They happen to have the optimal amount of x_2 already! For any other output however, having x_2 fixed at 16 will be inefficient and lead to higher costs. This is true in general.

y	Short-Run Cost	Long-Run Cost
1	16.0625	2
2	17	8
3	21.0625	18
4	32	32
5	55.0625	50
6	97	72
7	166.063	98
8	272	128

Figure 14.1: Comparison of Short- and Long-Run Costs

Info 14.5: Short-Run & Long-Run Costs. Short-run costs are always (weakly) higher than long-run costs.

14.8 Key Topics

- Understand that profit maximization implies cost minimization.
- Understand isocost lines ($w_1x_1 + w_2x_2 = c$). Calculate and interpret their slope.
- Understand isoquants ($f(x_1, x_2) = y$). Calculate and interpret their slope (technical rate of substitution).
- Understand that the cost minimizing input bundles occur where the isoquant "just touches" the isocost, but do not cross it.
- Understand and interpret the tangency condition and when to use it to find cost minimizing bundles.
- Understand how to find cost minimizing bundles for perfect complements production by using the "no-waste" condition.
- Understand how to find cost minimizing bundles for perfect substitutes production by checking whether it is cheaper to use only x_1 or only x_2 as in *Exercise 14.3*.
- Find the conditional factor demands x_1^* and x_2^* for a production function. These are the amount of inputs used to produce a given level of output at minimum cost as in *Exercises 14.1 - 14.4*.
- Using conditional factor demands, formulate the cost function $c(y) = w_1x_1^* + w_2x_2^*$, representing the minimum cost required to produce output y as in *Exercises 14.1 - 14.4*.
- Calculate the marginal cost from the cost function and categorize it into increasing, decreasing, or constant marginal cost as in *Exercise 14.1*.

- Understand the difference between short-run and long-run and be able to calculate short-run cost functions as in *Exercise 14.4*.
- Know that short run costs are always (weakly) higher than long-run costs as in *Exercise 14.4*.

15 Profit Maximization: Price-Taking

Once we have the cost function for a firm, we can write the profit function as a function of y by replacing $w_1x_1 + w_2x_2$ with $c(y)$ to get $\pi(y) = p(y)y - c(y)$. Since $c(y)$ is the cheapest way of producing y, this will give the most profit a firm could possible earn if it produces output y. The firm is just left to chose the optimal y. This is very easy to maximize since it is just one-dimensional. It only depends on y.

Definition 15.1: Profit in terms of y (output). The profit function in terms of output y is $\pi(y) = yp(y) - c(y)$ where $p(y)$ is the function that determines how price reacts to the firm's output and $c(y)$ is the firm's cost function.

We still need to know what $p(y)$ is. But for now, let's use a simple assumption that price does not depend on output y, the firm just assumes the price they will get for each unit of output is fixed at p. This is called the price-taking assumption.

Definition 15.2: Price-Taking. The price-taking assumption is that $p(y) = p$. That is, the price a firm gets for each unit of output is fixed at p and does not depend on their output.

This assumption is not valid in many cases. The idea that the price a firm can get for *any* amount of output it chooses is unreasonable. But if the firm is a *very small* part of a market (we call this **perfect competition**) it is probably an assumption we can get away with. We will discuss this more in class.

In any case, if we make the price-taking assumption, we can write profit as $\pi(y) = py - c(y)$.

Definition 15.3: Profit Under Price-Taking. The profit function of a firm under the price-taking assumption is $\pi(y) = py - c(y)$.

Let's look at the example from above. If we want to maximize profit with the production function f$(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$ and the price of output is assumed to be fixed at p, profit is $\pi(y) = py - 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2$. Notice we have plugged in the cost function we found above.

For an interior maximum (y is some number other than 0), the slope of this will have to be zero at the optimum y^*. Otherwise, the firm could increase or decrease output and increase profit. The first order condition is: $\frac{\partial(\pi(y))}{\partial y} = 0$ which here is $p - 4\sqrt{w_1}\sqrt{w_2}y = 0$. Notice, we can rewrite this as: $p = 4\sqrt{w_1}\sqrt{w_2}y$. The left side of this is the extra revenue from increasing output by one unit (the marginal revenue (MR). The right side of this is the extra cost from increasing output by one unit (the marginal cost (MC). It will always be true that the firms optimal output solves where $MR = MC$.

Definition 15.4: Marginal Revenue. A firm's **marginal revenue** (MR) is how their revenue increases as output increases. It is the derivative of revenue with respect to output y.

Under the price taking assumption (that price p does not depend on y) the marginal revenue is just p and we have $p = MC$. Returning to the example, we can solve y to get the optimal y for any set of prices: $y^*(p, w_1, w_2) = \frac{p}{4\sqrt{w_1}\sqrt{w_2}}$.

This the optimal (profit maximizing) level of output for any price. We can also write the "profit function" take this optimal level of output and plugs it back into the "conditional profit function". We found previously that this conditional profit function is: $\pi(y) = py - 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2$.

Plugging in the optimal level of production yields the profit function: $\pi(y^*) = p\left(\frac{p}{4\sqrt{w_1}\sqrt{w_2}}\right) - 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} \left(\frac{p}{4\sqrt{w_1}\sqrt{w_2}}\right)^2 = \frac{p^2}{8\sqrt{w_1}\sqrt{w_2}}$. Suppose $p = 10$ and $w_1 = w_2 = 1$ the maximum profit the firm can earn is (plug prices into the profit function above): $\pi^* = \frac{100}{8} = \frac{25}{2}$. Find the optimal level of output by plugging prices into the optimal output function $y^*(p, w_1, w_2) = \frac{p}{4\sqrt{w_1}\sqrt{w_2}}$. $y^* = \frac{10}{4} = \frac{5}{2}$.

15.1 More on Supply Under Price-Taking

When we make the price-taking assumption (firms take prices as fixed at p), firms do not consider how their quantity chosen affects the market price. Their profit function is $\pi = py - c(y)$. Maximizing this requires finding the point where the slope of the profit function is zero. This occurs where: $p = \frac{\partial(c(y))}{\partial y}$ or $p = mc(y)$.

For example, suppose $c(y) = 5y^2$. Marginal cost (MC) is $10y$. Setting $p = mc(y)$ we have $p = 10y$. This is the firm's **inverse supply**. It says "what price is responsible for the firm producing y units of output?" Thus, in a sense, the firm's marginal cost function *is* their inverse supply. We can invert it to get the supply. In this case, $y = \frac{p}{10}$.

15.2 What can go wrong– Linear/Increasing Returns to Scale

If returns to scale are linear or increasing, then if we can find any output level y where the firm earns positive profit, then there is no profit maximizing level of y. The firm wants to produce as much as possible. There is not optimal level of y. This is because with linear or increasing returns to scale, doubling inputs will double cost and at least double output- so profit will at least double. Thus, if we can find a point were profit is positive, we can always use of all inputs and increase profit.

Let's see this in an example. Suppose $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. Price of output is $p = 100$ and $w_1 = 1$, $w_2 = 1$. In this case, the cost minimizing level of inputs are (try this yourself using cost minimization): $x_1 = x_2 = y$. The cost function is: $c(y) = 2y$. The profit function in terms of y is: $\pi(y) = 100y - 2y = 98y$. This profit function is increasing in y. The more the firm produces, the more they will make in profit. In this case, there is no profit maximizing solution! They just want to produce more and more.

As another example, suppose you want to maximize profit using production function $f(x_1, x_2) = min\left\{\frac{1}{2}x_1, x_2\right\}$. To minimize costs, the firm should use: $\frac{1}{2}x_1 = x_2$. Plug this back into the production function to get the conditional factor demands: $x_1 = 2y$ and $x_2 = y$. The cost function is $c(y) = (2w_1 + w_2)y$.

The conditional profit function is: $\pi(y) = py - (2w_1 + w_2)y = (p - 2w_1 - w_2)y$. If $p > 2w_1 + w_2$ there is no profit maximizing level. I want to produce as much as possible. If $p < 2w_1 + w_2$ optimal level is $y = 0$ and profit is 0. If they are equal the profit is always zero and the firm can choose whatever they want.

15.3 Key Topics

- Using a firm's cost function $c(y)$, set up a firm's profit function in terms of y under price taking as in *Exercises 15.1 - 15.3*.
- Understand what the price-taking assumption is and when it is reasonable/unreasonable.
- Be able to find the optimal amount of output y^* (that maximizes profit) as in *Exercises 15.1, 15.3*.
- Understand where the profit maximization condition $p = mc(y)$ comes from and why it is specific to a scenario with price-taking.

16 Monopoly

16.1 Monopolies and the Price-Taking Assumption

Most firms cannot reasonably assume that price is fixed in their output. If Apple wants to sell a million more iPhones, they cannot hope to do it at the price they are currently selling iPhones. That is, the price p they will get depends on y (their output). How does p depend on y? That is, what is the function $p(y)$? In general, this depends on both the market demand for that good and the structure of competition in the market. Apple would need to lower prices to sell their one million more iPhones, but then other phone companies could also try lower their prices as well. In general, the way price depends on demand can be quite complex.

In the case of price-taking, we simplify things drastically by assuming price does not depend on output at all, but this is usually not very realistic. The next most-simple scenario is to assume away the competition, letting price depend on output in a way that is purely determined by **demand** and not competition. This is the case for monopolies which have no competition.

> **Definition 16.1: monopoly.** A monopoly is a firm that is the only seller of some good.

Of course, when a monopoly wants to sell y units of a good, they will try to sell it at the highest price they can. Since there is no competition, there is no reason to charge less than the maximum they can sell y units for. What is the maximum they can charge? It is whatever consumers will pay for y units- **that's the inverse market demand**!

> **Definition 16.2: Price and output for a monopolist..** The function $p(y)$ for a monopolist is the inverse market demand function.

For example, suppose market demand is: $\frac{\frac{1}{2}(200)}{p}$. The amount the firm can sell at price p is $y = \frac{\frac{1}{2}(200)}{p}$ Then inverse demand is $p = \frac{\frac{1}{2}(200)}{y} = \frac{100}{y}$. For example, the most this monopolist could charge to sell $y = 100$ units is $p = 1$. If the monopolist wants to sell $y = 200$, the most they could charge is $p = \frac{100}{200} = \frac{1}{2}$. As y increases, they amount they can charge will decrease.

16.2 The Monopolist's Profit Function

> **Definition 16.3: Profit Function of a Monopolist.** Let $p(y)$ be the inverse market demand and let $c(y)$ be their cost function. The **monopolist's profit function** profit function is: $\pi\left(y\right) = p\left(y\right)y - c\left(y\right)$.

Notice that the monopolists revenue is $p\left(y\right)y$ and the cost is $c\left(y\right)$. Contrast this with profit under the price taking assumption where revenue is $p(y)$ and cost is $c(y)$.

To maximize profit, we need to look for where marginal profit is zero. That is where $\frac{\partial p(y)y - c(y)}{\partial y} = 0$ or $\frac{\partial p(y)y}{\partial y} - \frac{\partial c(y)}{\partial y} = 0$. Simplifying further we get $\frac{\partial p(y)y}{\partial y} = \frac{\partial c(y)}{\partial y}$. That is where the marginal revenue is equal to marginal cost!

That is, at the optimum, the firm is still setting marginal revenue to marginal cost. Suppose that was not the case. If marginal revenue is higher, then the firm can increase revenue more than cost by increasing output. This will increase profit. If marginal cost is more than marginal revenue, decreasing output will increase profit by lowering cost more than revenue.

The key difference between this and the price taking assumption is that the marginal revenue is simply the price p under price taking, but for a monopolist, the marginal revenue is $\frac{\partial p(y)y}{\partial y}$. To simplify this, we can use the product rule. Letting $p'(y)$ be the derivative of $p(y)$ with respect to y, this marginal revenue is $p'(y)y + p(y)$. The new term is $p'(y)$.

For a monopolist, how does revenue change when they change output? Since they sell more, it goes up directly by the price they are charging $p(y)$ but then, for every unit they are selling, they have to charge less! Thus, the price also changes by $p'(y)y$! Since the inverse demand is downward sloping, $p'(y)$ is negative. Thus, while revenue goes up by $p(y)$, it goes down by $p'(y)y$. The monopolist needs to account for *both* of these effects when maximizing profit!

16.3 Example of Maximizing Profit

Suppose the demand for a good supplied by a monopolist is $y = 100 - p$ and their cost is $c(y) = 10y$.

To construct the monopolist's profit function, we first find the inverse demand function, which is $p = 100 - y$. Using this, we can construct the firm's profit function:

$$\pi(y) = (100 - y)\, y - 10y$$

To maximize this, find where it's derivative is zero. $\frac{\partial((100-y)y-10y)}{\partial y} = 0$ which is $90 - 2y = 0$.

Solving this for y gives us the optimal level of output: $y = 45$. Plugging this into the inverse demand function gives us the highest price the monopolist can sell these units for: $p = 55$.

We can now calculate the firm's profit: $\pi(45) = (55)(45) - 10(45) = 2025$.

16.4 What does a monopoly do?

A monopolist's quantity decision depends a lot on the consumer demand. A monopolist leverages their market power by raising prices. Of course, to do this, they have to restrict quantity. If demand is inelastic, raising price by 1 percent lowers demand by less than one percent. Looking at the the other way around, lowering quantity by one percent allows a monopolist to raise price by more than one percent. This will lead to an increase in revenue since price increases proportionally more than quantity decrease. At the same time, lowering quantity will also lower costs.

In summary, when demand in inelastic, the monopolist can lower quantity, increase revenue and also decrease costs. This has to increase profit!

This tells us that if a monopolist is acting optimally, they will always continue lowering quantity as long as demand remains inelastic. Thus, any profit maximizing quantity can only occur where demand is elastic.

> **Info 16.1: Monopoly and Elastic Demand.** At the monopolist's profit maximizing quantity (assuming one exists), demand must be elastic.

16.5 Consumer Surplus Under Monopoly

As when we studied equilibrium, the consumer surplus under a monopoly is still the difference between what consumers are willing to pay for goods and what they actually pay. That is, the consumer surplus will still be measured by the area *under* the inverse demand curve but *above* price. For equilibrium, we motivated the producer surplus as the area *above* the inverse supply (what they are willing to accept) but also *above* price. We have not really calculated an inverse supply function for a monopolist. However, when any firm produces an extra unit of a good, their additional costs is their marginal cost. As long as they can sell that unit for their marginal cost of producing it, it is worth producing. In this sense, an individual firms lowest willingness to accept for selling a unit of a good is their marginal cost. The difference between price and marginal cost represents surplus for the firm. This way, we can motivate measuring the producer surplus of a monopolist as the area above their marginal cost curve but above price!

In our running example with a monopolist with cost function $c(y) = 10y$, the marginal cost is 10. Inverse demand is $100 - y$ (where y is the monopolist quantity). Let's draw these graphs.

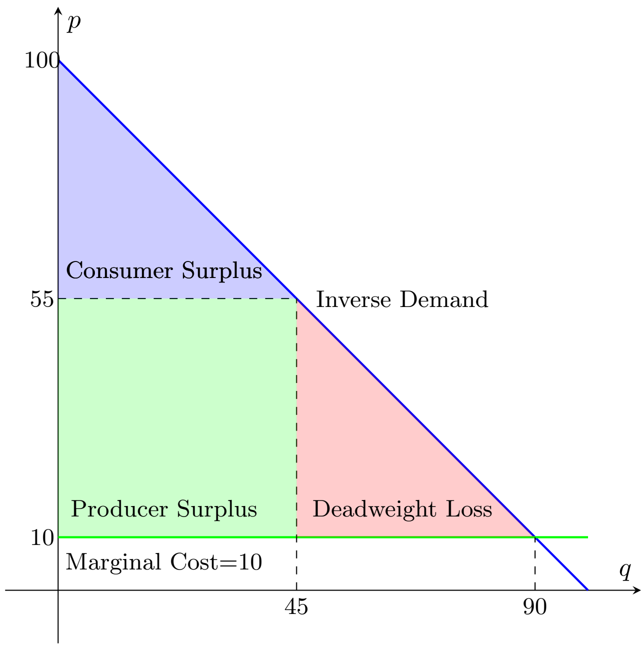


Figure 15: Monopoly Profit, Consumer Surplus, and Deadweight Loss.

The consumer surplus is the area below the inverse demand of $p = 100 - y$ but above the price of 55. This is the blue triangle in the figure above. It has an area of $\frac{45*(100-55)}{2} = 1012.5$.

The monopolist's producer surplus is the area above their marginal cost 10 but below the price they get 55. This is the green triangle and has an area of $45(55 - 10) = 2025$

Notice that if the monopolist was willing to sell the good at the lowest price they would accept, (their marginal cost) 10, they could sell 90 units. Of course, the monopolist would enjoy *zero* surplus in this case, but the entire triangle made up of the blue, green, and red areas would be captured as consumer surplus. That is actually the most total surplus that could be generated in this example. Relative to that scenario, the red triangle is not captured by anyone and is left as deadweight loss. It has an area of $(55-10)(90-45) = 2025$.

Thus, while a monopolist tries to use their market power to capture as much profit/surplus as possible, we can see here they leave some surplus on the table. Some of the potential surplus they could get is left with the consumers (blue area) and some is not captured by anyone (red area).

However, monopolists have some tricks up their sleeves to capture more surplus, but to do it they will need to charge different people, different prices. We will look at that in the next chapter.

16.6 Key Topics

- Be able to set up a monopolist profit function from the market demand and cost function as in *Exercises 16.1 -16.4*.
- Be able to solve for the profit maximizing output for monopolists as in *Exercises 16.1 -16.4*.
- (*Conceptual*) Understand that monopolists take advantage of their market power by restricting output to drive up the price.
- Understand why a monopolists optimal output never occurs where demand is inelastic.
- Be able to calculate the consumer surplus, producer surplus, and deadweight loss under monopoly when market demand is linear and there are constant marginal costs as in *Exercises 16.3, 16.4*.

17 Price Discrimination

In the pervious chapter, we looked at what a monopolist can achieve by leveraging their market power to create scarcity and drive up the price in order to improve profit. However, as seen in figure 15, the monopolist ends up leaving a lot of surplus on the table both in terms of remaining consumer surplus (charging some consumers less than they are willing to pay) and dead-weight loss (lost surplus due to excluding some consumers from the market through the artificial scarcity). In this chapter, we look at more complex ways a monopolist can sell their products to try and capture some of this surplus.

17.1 Types of Price Discrimination

There are several ways we will look at for monopolists to price their goods to get more profit. Here is a brief list:

- **First Degree Price Discrimination**: The firm can identify every consumer, learn their willingness to pay, and charge different price.
  - This is an extreme form of price discrimination. It is best understood as a thought experiment about the extreme of what a monopolist could do rather than something actually achievable.
  - Examples: Airlines sometimes come close to this when they use complex pricing schemes to try and extract more and more surplus from consumers. Everyone on the airplane probably paid a different price for their tickets, but it is unlikely they all paid their highest willingness.
- Second Degree Price Discrimination: The firm cannot identify individual consumers, but can offer different packages or qualities of goods at different prices.
  - Examples: Quantity discounts, quality differences (first-class/coach tickets, "reserve" wines, "flagship" high-end products that differ little from cheaper counterparts.
- Third Degree Price Discrimination: Can identify groups and charge those groups different amounts.
  - Examples: Student tickets. Senior discounts.
- Bundling: Combine different goods and force consumers to buy them in bundles.
  - Examples: Cable TV Packages, Microsoft Office Software Bundle.
- Two-Part Tariff: Charge the consumer and entry fee or membership that give the consumer the right to buy the good at the lowest efficient price (the marginal cost of the firm).
  - Examples: Netflix (compare this to streaming rental services), Theme park tickets (rides are free), free-coffee for the month when you pay $19.99 to buy a special mug.

17.2 First Degree Price Discrimination

> **Definition 17.1: First-Degree Price Discrimination.** In **first-degree** price discrimination, the firm charges every consumer their full willingness-to-pay.

For example, suppose there are three people willing to pay \$3, \$2, \$1 for a good respectively. Suppose the monopolist has zero marginal cost. Here is the monopoly profit at different prices **if it charges everyone the same price**:

Price	# Buyers	Profit
\$3	1	\$3
\$2	2	\$4
\$1	3	\$3

The most the firm can make is \$4. But if it knew everyone's willingness to pay and could charge them different prices, charging one buyer \$3, one buyer \$2, and one buyer \$1. If it does this, the firm could earn \$6!

When the monopolist charges one price, the consumers get some consumer surplus (refer to Figure 15) This is because there are some consumers who get the good at a price lower then they are willing to pay. On other hand, there is also some dead weight loss because the monopolist restricts quantity from the efficient level where price is equal to marginal cost. When a monopolist uses first degree price discrimination there is no dead-weight loss and they capture all of the consumer surplus! This is because the firm can sell to everyone who it is efficient to sell to (they are willing to pay more than marginal cost) at exactly the price they are willing to pay.

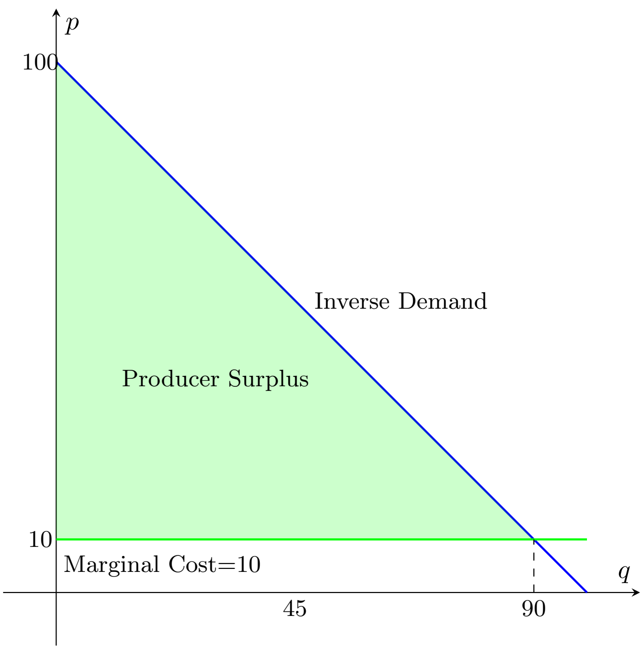


Figure 16: Monopoly Surplus under First-Degree Price Discrimination.

17.3 Third Degree Price Discrimination in Action

Definition 17.2: Third-Degree Price Discrimination. In **third-degree price discrimination**, the monopolist can identify different groups of people that have different demand. It treats the different groups separately, and **maximizes the profit within each group -** charging them each a different price.

For example, suppose there are two groups of people: students and non-students. A movie theater sells tickets to both groups. Assume the firm has zero marginal cost so that $c(y) = 0$ (cost is zero regardless of output). Students have demand function: $y_s = 100 - 2p$ and non-students have demand function: $y_n = 100 - p$.

If we add up both types of consumer, entire market demand is: $Y = 100 - 2p + 100 - p = 200 - 3p$ (as long as $p \leq 50$). The inverse demands for both groups, and the market as a whole are $p_s = \frac{100-y_s}{2}, p_n = 100 - y_n, p = \frac{200-Y}{3}$.

Suppose the monopolist was going to set one price for the entire market. Their profit function would be: $\pi = \frac{200-Y}{3}Y$. By taking the first order condition and solving we find that the optimal $Y = 100$ and the optimal price is $p = \frac{100}{3}$. At this price $y_s = \frac{1}{3}(100)$ (about 33) is the student demand and $y_n = \frac{2}{3}(100)$ about 66 is the non-student demand. The firm's profit is: $\pi \approx 3333.33$.

What if the firm wanted to set prices differently for students and non-students?

The profit earned from students is: $\pi_s = \frac{100-y_s}{2}y_s$. The profit earned from non-students is: $\pi_n = (100 - y_n)\,y_n$. Solving the first-order conditions, we get that the optimal $y_s = 50$ and the optimal $y_n = 50$. The prices the firm can charge are $p_s = 25$ and $p_n = 50$. The profits are: $\pi_s = 1250$ and $\pi_n = 2500$.

The total profit is: $\pi = \pi_s + \pi_n = 3750$. Notice the firm can earn about 416.67 more by setting different prices!

The key insight here is that by treating each group separately, the monopolist can extract more surplus from each group based on their specific demand characteristics. This works because the groups have different price elasticities of demand - students are more price sensitive (elastic demand) while non-students are less price sensitive (inelastic demand). The monopolist can thus charge higher prices to the less elastic group while maintaining sales to the more elastic group.

17.4 Bundling

Definition 17.3: Bundling. Bundling occurs when a firm sells multiple products together as a package at a single price, rather than selling each good separately. This can be profitable when consumers have different valuations for different goods, and the firm can capture more surplus by forcing consumers to purchase goods together.

Bundling can occur when a firm sells multiple products. The goal of bundling is to take advantage of differences in types of demand by forcing consumers to buy bundles of goods at a single price rather than selling each good at a separate price.

For Example, suppose a firm sells pants and shirts. There are two consumers who each demand up to one shirt and one pair of pants. They are willing to pay the following:

	Shirt	Pants	Both
Consumer 1	50	30	80
Consumer 2	10	80	90

Pricing Shirts.

If they price shirts at \$50, they sell one shirt and earn \$50. If they price at 10, they sell two shirts and earn \$20.

Pricing Pants.

If they price pants at \$80, they sell one pair of pants and earn \$80. If they price at 30, they sell two pairs of pants and earn \$60.

Thus, the best they can do is sell one shirt at \$50 and one pair of pants at \$80 to earn \$130.

Pricing Bundles.

If the firm forces consumers to buy a bundle of a shirt and a pair of pants they can price that bundle at \$80, sell two bundles and earn \$160.

The key to understanding why bundling works in this example is that the consumers have negatively correlated valuations - Consumer 1 values shirts highly but pants less, while Consumer 2 values pants highly but shirts less. By bundling, the firm can capture more of the total willingness to pay from both consumers. This is particularly effective when the sum of individual valuations for the bundle is more similar across consumers than their individual valuations for each good.

17.5 Two-Part Tariff

> **Definition 17.4: Two-Part Tariff.** A **two-part tariff** is a pricing strategy where a firm charges consumers both a fixed fee (entry fee) and a per-unit price. The goal is to set the per-unit price at marginal cost to maximize total surplus, then capture that entire surplus through the fixed fee.

Two-part tariffs can be used when consumers demand multiple units of a good. An example of this is theme parks tickets. The theme park could charge a price per ride. In fact, this happens at some fairs. However, instead, rides are free once you have purchased the ticket. The goal of a a two-part tariff is to create as much consumer surplus as possible by selling the consumer as much as is efficient (this occurs where price is marginal cost). This will create the most consumer surplus possible. Instead of leaving that consumer with the surplus, charge them an "entry fee" (this is the other part of the tariff) equal to their consumer surplus.

For Example, suppose a consumer's demand for coffee is $q = 10 - p$ and the firm has zero marginal cost for coffee. If the firm sells to that consumer at a single price it's profit of selling the consumer q cups of coffee at the most they will pay for those q cups is: $\pi = (10 - q)\,q$. The best thing to do is sell them 5 cups of coffee at 5 dollars and earn \$25.

If the firm prices at marginal cost (\$0) the consumer will demand 10 cups of coffee. Their surplus is the area below the inverse demand but above price of zero. That surplus is \$50, so they would be willing to pay up to \$50 for the right to buy cups of coffee at \$0 (assuming you don't give them the option to buy at \$5 per cup). So the firm can earn \$50 by forcing the consumer to pay an "entry fee" of \$50 and then give them coffee for free.

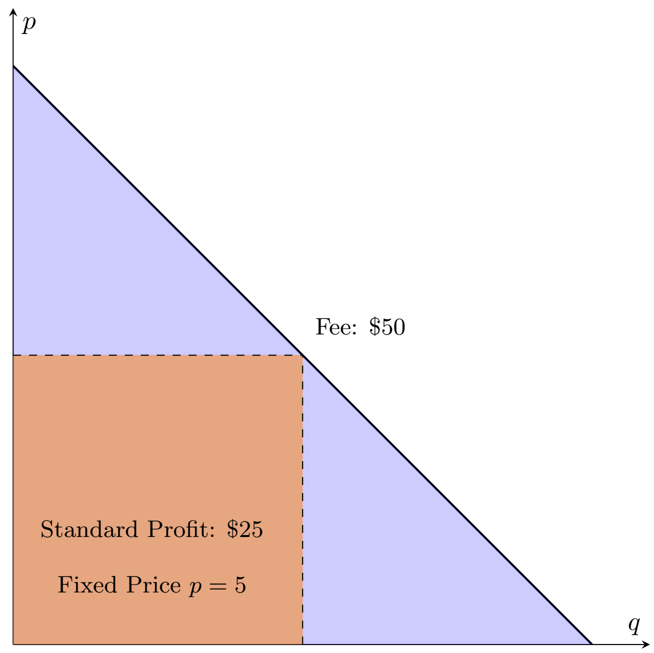


Figure 17: Earning more with a two-part tariff. Orange is the profit under optimal unit pricing. Blue plus orange area is the consumer surplus under marginal cost pricing that is then captured with an entry fee.

The two-part tariff is particularly effective when consumers have similar demand curves but different quantities demanded. By setting the per-unit price at marginal cost, the firm ensures that each consumer purchases the surplus-maximizing quantity. The fixed fee can then then be used to capture the the entire consumer surplus that would otherwise be lost under standard monopoly pricing. This strategy is commonly used in membership-based services, theme parks, and subscription services where the marginal cost of serving additional units is relatively low.

17.6 Key Topics

- Understand the different types of price discrimination (first-degree, second-degree, third-degree, bundling, two-part tariff) and be able to identify examples of each as in *Exercise 17.1.*
- Be able to calculate optimal prices and quantities for two groups under third-degree price discrimination when given demand functions for different groups and compare this to the profit when a firm uses standard monopoly pricing treating both groups as a single market as in *Exercises 17.2, 17.3.*
- Understand how bundling works and why it can be profitable and solve bundling problems to find the optimal bundle price in simple examples with two goods and two consumers

and compare the profit from bundling to the profit from selling the goods separately at different prices as in *Exercises 17.4, 17.5*.

- Know how to set up and solve a two-part tariff problem given a demand function for each consumer and a fixed marginal cost. Find the optimal per-unit price (marginal cost) and entry fee to maximize profit as in *Exercises 17.6, 17.7*.

18 The Cournot Model of Competition

18.1 Extending the Monopoly Model

To relax the price-taking assumption, we need to know the relationship between a firm's quantity and the market price. For a monopoly, that relationship is easy to figure out, it is the inverse demand. The inverse demand is the *most* that consumers will pay to buy q units of the good. So, if a monopolist produces q units of the good, they can charge the inverse demand $p(q)$ at that quantity.

But what about when there are multiple firms? The inverse demand represents the relationship between the price and the market quantity. But when there are many firms, each firm's quantity is only a part of the market quantity. In the Cournot model of oligopoly, we assume firms compete with each-other only through the quantity the choose.

> **Definition 18.1: Cournot Model.** In the **Cournot model** of "oligopoly" (competition between several firms), the firms compete only through the quantity they choose. Price is determined by what consumers will pay for the resulting market quantity.

The first set to extending the model is to make sure we can keep track of each firm's quantity, and the total market quantity. We use the following notation:

> **Definition 18.2: Quantities in a Cournot Model.**
>
> **Individual firm quantity:** q_i: firm i's quantity
> **Market Quantity**: Q: $Q = \sum_{i=1}^{n} q_i$
> **Total "Other" Firm's Quantities**: Q_{-i}: $Q_{-i} = Q - q_i$. This is the total quantity of all firms except i.

We know that the amount consumers will pay for the **market quantity** is the inverse demand $p(Q)$.

Thus, the profit of a firm is dependent both on the quantity they produce and the total market quantity. The price the firms get is $p(Q)$. Firm i sells q_i units. Thus, their revenue is $q_i p(Q)$ and their cost is $c(q_i$. This lets us write each firm's profit function like this:

$$q_i p(Q) - c(q_i)$$

.

Notice here, the market price is determined by the total market quantity. Revenue is that price multiplied by the firms own quantity. Of course, costs are specific to a firm and dependent only on their own quantity. One issue with writing the profit this way is that Q itself depends on q_i. It is more convenient to write the profit function in terms of q_i and Q_{-i}. Noting that $Q = Q_{-i} + q_i$, we can write: $\pi(q_i, Q_{-i}) = p(Q_{-i} + q_i) q_i - c(q_i)$.

> **Definition 18.3: Profit Function of a Firm in Cournot Model.** The profit of firm i in the Cournot model is:
> $\pi(q_i, Q_{-i}) = p(Q_{-i} + q_i) q_i - c(q_i)$

If the firm knew (or even had some assumption about) what Q_{-i} is, they could calculate their profit for any quantity q_i they produce and even maximize it. Let's look at an example.

18.2 Example of Maximizing Profit with Two Firms

Suppose inverse demand is $p(Q) = 100 - Q$, there are two firms, and the cost function of each firm is $c(q_i) = 10q_i$.

Firm 1's profit function is:

$$\pi_1(q_1, q_2) = (100 - q_1 - q_2) q_1 - 10q_1$$

This simplifies to:

$$\pi_1(q_1, q_2) = 90q_1 - q_i^2 - q_1 q_2$$

Similarly 2's profit function is

$$\pi_2(q_1, q_2) = 90q_2 - q_2^2 - q_1 q_2$$

18.2.1 Game Theory

This model is a **Game**. A game is a formal mathematical object studied in game theory.

> **Definition 18.4: Game.** A **game** is a set a set of **players**, a set of actions called **strategies** for each player, and a set of **payoffs** for each player that may depend on the strategies chosen by the players.

There are many ways to "solve" a game. That is, to make predictions about what might happen in that game given the strategic sophistication of players in the game. The most common way to solve a game in game theory is to use the **Nash equilibrium.** To define a Nash Equilibrium, let's first look at **best responses.**

18.2.2 Nash Equilibrium for 2 Firms

Suppose, in our example, firm 1 believes firm 2 will produce $q_2 = 50$. Then they believe their profit function is $\pi_1(q_1, 50) = 40q_1 - q_1^2$. To maximize this, find where the derivative is zero. This occurs at $q_1 = 20$.

In game theory, $q_1 = 20$ is what we call a **best response** to $q_2 = 50$.

> **Definition 18.5: Best Response.** In game theory, a **best response** is the optimal strategy for a player, conditional on the strategies chosen by the other players.

When a set of quantities are *simultaneously* best responses to each other, we say that are a **Nash Equilibrium** of the game. Since Nash Equilibrium requires all players to be simultaneously best responding to each other, it will also imply that no player has incentive to change their action.

While $q_1 = 20$ is a best response to $q_2 = 50$. Firm 2's profit if firm 1 produces $q_1 = 20$ is: $\pi_2(20, q_2) = 90q_2 - q_2^2 - 20q_2$ which is maximized at 35. Thus, $q_2 = 35$ is the best response to

$q_1 = 50$ and so the pair $(20, 50)$ is **not** a Nash equilibrium since firm 2 is not best responding in choosing $q_2 = 50$.

To find the Nash equilibrium, let's look again at each firm's profit function:

$$\pi_1(q_1, q_2) = (100 - q_1 - q_2)\, q_1 - 10q_1 = 90q_1 - q_1^2 - q_1 q_2$$

$$\pi_2(q_2, q_1) = (100 - q_1 - q_2)\, q_2 - 10q_2 = 90q_2 - q_2^2 - q_2 q_1$$

To find firm 1's best response to any q_2, maximize their profit by finding where their marginal profit is zero (the first-order condition). Firm $1's$ marginal profit is:

$$\frac{\partial\left(90q_1 - q_1^2 - q_1 q_2\right)}{\partial q_1} = 90 - 2q_1 - q_2$$

It's marginal profit is zero, and it's profit is maximized where when

$$q_1 = \frac{90 - q_2}{2}$$

This is firm 1's **best response function**. It tells firm 1 what quantity to choose given what firm 2 chooses.

> **Definition 18.6: Best Response Function.** A **best response function** is a function that tells a player what strategy to choose given the strategies chosen by the other players. In the Cournot model, the best response function is the quantity a firm should choose given the quantity chosen by the other firms.

Similarly, firm 2's marginal profit is:

$$\frac{\partial\left(90q_2 - q_2^2 - q_1 q_2\right)}{\partial q_1} = 90 - 2q_2 - q_1$$

likewise $q_2 = \frac{90 - q_1}{2}$.

Solving where firm 2's marginal profit is zero gives us firm 2's profit maximizing choice (their best response) to any choice by firm 1.

$$q_2 = \frac{90 - q_1}{2}$$

A **Nash Equilibrium** is a pair (q_1, q_2) that simultaneously solves both of these best response functions.

This system of equations, $\left\{q_1 = \frac{90-q_2}{2}, q_2 = \frac{90-q_1}{2}\right\}$, has only one solution. This is the **Nash equilibrium** of the game:

$$q_1 = 30, q_2 = 30$$

Notice how, the Nash equilibrium here is symmetric. That is, both firms choose the same quantity. Any Cournot model with firms that have the same cost function will have a symmetric Nash equilibrium. In fact, the equilibrium in such a symmetric Cournot model will be symmetric!

Info 18.1: Symmetric Nash Equilibrium in Symmetric Cournot Models. In a Cournot model *if all firms have the same cost function*, the Nash equilibrium will be symmetric.

This fact gives us a shortcut to finding the Nash equilibrium in a symmetric Cournot model. Let's look at the example again. The best response functions are:

$$q_1 = \frac{90 - q_2}{2}$$

$$q_2 = \frac{90 - q_1}{2}$$

If we assume that both firms will choose the same quantity, we can substitute $q_1 = q_2 = q$ into the best response functions to solve for q:

The best response functions both become:

$$q = \frac{90 - q}{2}$$

Solving this for q gives us:

$$2q = 90 - q$$

$$q^* = 30$$

This is called **imposing symmetry**.

Definition 18.7: Imposing Symmetry. Imposing symmetry is a method for finding the Nash equilibrium in a symmetric Cournot model. To impose symmetry, assume $q_i = q_j = q$ for all firms. Plug this into the best response functions and solve for q.

18.3 Equilibrium with N firms.

Now suppose we have N firms in the same model instead of two firms. Recall that $Q = q_i + Q_{-i}$. Thus, for the profit function of firm i, we have:

$$\pi_i\left(q_i, Q_{-i}\right) = q_i\left(100 - q_i - Q_{-i}\right) - 10q_i$$

This simplifies to:

$$\pi_i(q_i, Q_{-i}) = 90q_i - q_i^2 - q_i Q_{-i}$$

Here, our goal is to find firm i's best response function to the total quantity of the other firms Q_{-i}. That is, the quantity firm i should choose given the quantities chosen by the other firms. To do this, we need to find where the marginal profit of firm i is zero. This occurs where $\frac{\partial(90q_i - q_i^2 - q_i Q_{-i})}{\partial q_i} = 90 - 2q_i - Q_{-i} = 0$. Solving this for q_i gives firm i's best response function to Q_{-i} by the other firms

$$q_i = \frac{90 - Q_{-i}}{2}$$

Normally, to find the Nash equilibrium, we would need to solve N equations of the form $q_i = \frac{90-Q_{-i}}{2}$ (one for each firm) for the n unknowns (one for each q_i). However, notice each firm's best response function is effectively identical because they all have **the same cost function**. This is this is a symmetric model (same cost functions). Let's impose symmetry for solve for the Nash equilibrium.

First, note that if all firms choose the same quantity q, then $Q_{-i} = (N-1)q$ since there are $(N-1)$ firms that aren't firm i. Imposing symmetry, we get the equation: $q = \frac{90-((N-1)q)}{2}$.

Solving this for q gives us $q^* = \frac{90}{N+1}$. Thus, in equilibrium, with N firms, all will produce $q = \frac{90}{N+1}$. The market quantity in equilibrium will be $Q = (N)q^* = \frac{N}{N+1}90$ and the market price will be. $p^* = 100 - \frac{N}{N+1}90$. Let's look at this market price and market quantity as the number of firms changes:

N	Price: p	Individual Quantity q	Market Quantity: Q	Profit: π_i
1	55	45	45	2025
2	40	30	60	900
5	25	15	75	225
100	10.9	0.9	89.1	0.79
1000	10.1	0.09	89.9	0.008

Figure 18: Price and quantity in Nash equilibrium of the Cournot model with N firms when each has cost $c(q) = 10q$ and inverse demand is $p = 100 - Q$.

Notice that, for $N = 1$ we get the monopoly solution $p = 55$ $q = 45$! As N increases, the price approaches 10, the marginal cost of each firm and each firm's profit approaches 0. Each firm's market share also approaches 0, that is the quantity each firm relative to the entire market shrinks to 0 and the market power of each firm disappears, leading to perfect competition in the limit. In this sense, the Cournot model is a generalization of the both our monopoly model ($N = 1$) and perfect competition / price taking models ($N \to \infty$)!

18.4 Key Topics

- Understand the Cournot model of oligopoly competition and how it differs from monopoly and price taking models.

- Be able to write down a firm's profit function in the Cournot model, understanding how it depends on both the firm's own quantity and the quantities of other firms as in *Exercises 18.1 - 18.2*.
- Understand the concept of best response functions and how to derive them from a firm's profit function.
- Understand how to use symmetry to find the Nash equilibrium in a symmetric Cournot model with two firms as in *Exercises 18.1 - 18.2*.
- (**NOT ON EXAM**) Be able to solve for the Nash equilibrium in a Cournot model with N firms and understand how the equilibrium changes as N increases.

Part I

Exercises

19 Exercises

19.1 Exercises for Chapter 1

Exercise 1.1: How many units of x_1 and x_2 are in the bundle $(3, 4)$?

Exercise 1.2: List three bundles in the budget set $x_1 + x_2 \leq 5$

Exercise 1.3: Sketch the Budget set B that consists of bundles where $x_1 + x_2 \leq 5$.

Exercise 1.4: Qualitatively, if a tax is imposed on x_1 what does this do to the trade-off between x_1 and x_2 along th budget line?

For exercises **1.5-1.14**, assume income is $m = 10$ and prices $p_1 = 1$, $p_2 = 2$.

Exercise 1.5: Is $(2, 3)$ in the budget set? Is it on the budget line?

Exercise 1.6: Is $(1, 5)$ in the budget set? Is it on the budget line?

Exercise 1.7: Is $(2, 4)$ in the budget set? Is it on the budget line?

Exercise 1.8: Write down the equation of the budget line.

Exercise 1.9: How much x_1 can the consumer afford if they only buy x_1? How about x_2?

Exercise 1.10: What is the slope of the budget line?

Exercise 1.11: Sketch the budget line. Label the slope and endpoints.

Exercise 1.12: Plot the budget line again, labeling the slope and endpoints. Demonstrate what happens when m increases to 20. Label the slope and endpoints of the new budget line as well.

Exercise 1.13: Plot the budget line again, labeling the slope and endpoints. Demonstrate what happens when p_1 increases to 2. Label the slope and endpoints of the new budget line as well.

Exercise 1.14: Write down the equation for the budget line if a quantity tax of $t = 2$ is imposed on x_2.

19.2 Exercises for Chapter 2

For the following three questions, answer "yes" or "no" for each property.

Exercise 2.1: For the following exercise, use whatever definition you want. This exercise is meant to show you how the particular definition of something can affect its formal properties. Is the relation "is a sibling of" on the set of all people:

a. reflexive

b. complete

c. transitive

d. symmetric

e. asymmetric

Exercise 2.2: Is the relation "is at least as tall as" on the set of all people:

a. reflexive

b. complete

c. transitive

d. symmetric

e. asymmetric

Exercise 2.3: Is the relation "has same birthday as" on the set of all people:

a. reflexive

b. complete

c. transitive

d. symmetric

e. asymmetric

For the relations R below, when a pair is not listed, assume that the relation is not true of that pair.

Exercise 2.4: For the set $X = \{x, y, z\}$, identify if the following relations are transitive. If a pair does not appear, you can assume the relation is not true for that pair.

a. $R : xRy, yRz, xRz$

b. $R : xRx, yRy, zRz$

c. $R : xRy, yRz, zRx$

Exercise 2.5: For the set $X = \{p, q, r\}$, identify if the following relations are complete and transitive. When a relation is not both of these, say which assumption fails and why. If a pair does not appear, you can assume the relation is not true for that pair.

a. $R : pRp, qRq, rRr, pRq, qRr$

b. $R : pRp, qRq, rRr, pRq, qRr, pRr$

c. $R : pRp, qRq, rRr, pRq, qRp, qRr, rRq, pRr, rRp$

d. $R : pRp, qRq, rRr, pRq, qRp, pRr$

Exercise 2.6: For the following relations on the set of numbers, determine which of the following properties hold: *reflexive, complete, transitive, symmetric, asymmetric*?

a. $=$

b. $>$

c. $\geq$

Exercise 2.7: **Harder:** Argue that a relation that is complete and symmetric is trivial in the sense that it relates all pairs to each other.

19.3 Exercises for Chapter 3

Exercise 3.1: Consider the preference relation that describes someone's preferences over left l and right r shoes, where they only care about the number of usable pairs of shoes they consume. Sketch the indifference curves $\sim (1, 1)$ and $\sim (2, 2)$ on graph that has l on the x-axis and r on the y-axis. Label the set $\succ (2, 2)$.

Exercise 3.2: Consider the preference relation that describes someone's preferences for red apples r and green apples g, where they only care about the total number of apples they have but not the color. Sketch the indifference curves $\sim (1, 1)$ and $\sim (2, 2)$ on graph that has r on the x-axis and g on the y-axis. Label the set $\succ (2, 2)$.

Exercise 3.3: Write the following preference relations in **chain notation**.

- $a \succsim b, a \succsim c, b \succsim a, b \succsim c, c \succsim a, c \succsim b, a \succsim a, b \succsim b, c \succsim c$

- $a \succsim b, a \succsim c, b \succsim a, b \succsim c, a \succsim a, b \succsim b, c \succsim c$
- $a \succsim b, a \succsim c, a \succsim d, b \succsim c, b \succsim d, c \succsim b, c \succsim d, a \succsim a, b \succsim b, c \succsim c, d \succsim d$

Exercise 3.4: Write the strict preference relation $\succ$ induced by each of the following weak preference relations:

a. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim r, p \succsim r$

b. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim p, q \succsim r, r \succsim q, p \succsim r, r \succsim p$

Exercise 3.5: Write the indifference relation $\sim$ induced by each of the following weak preference relations:

a. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim r, p \succsim r$

b. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim p, q \succsim r, r \succsim q, p \succsim r, r \succsim p$

Exercise 3.6: Consider the following preference relation on the set $\{a, b, c\}$:

$$a \succsim a, b \succsim b, c \succsim c, a \succsim b, b \succsim c, c \succsim a$$

- Is it complete?
- Is it transitive?

Exercise 3.7: Consider the following preference relation on the set $\{a, b, c\}$:

$$a \succsim a, b \succsim b, c \succsim c, a \succsim b, b \succsim a, b \succsim c, c \succsim b, a \succsim c, c \succsim a$$

- Is it complete?
- Is it transitive?

Exercise 3.8: Consider the following preference relation on the set $\{a, b, c\}$:

$$a \succsim a, b \succsim b, c \succsim c, b \succsim c, a \succsim c, c \succsim a$$

- Is it complete?
- Is it transitive?

Exercise 3.9: For the rational preference relation you wrote the chain notation for in Exercise 3.3:

$$a \succsim b, a \succsim c, a \succsim d, b \succsim c, b \succsim d, c \succsim b, c \succsim d, a \succsim a, b \succsim b, c \succsim c, d \succsim d$$

- What is **best** from set $\{a, b, c, d\}$?
- What is **best** from set $\{b, c, d\}$?
- What is **best** from set $\{c, d\}$?

19.4 Exercises for Chapter 4

Exercise 4.1: Consider bundles a, b, and c with the given utilities $U(a) = 8$, $U(b) = 15$, and $U(c) = 10$. What complete and transitive preference relation does this utility function represent (written in chain notation)?

Exercise 4.2: Provide an alternative utility function that represents the same preferences as those in the previous exercise.

Exercise 4.3: Suppose that a consumer's preferences can be represented by the utility function $u(x_1, x_2) = \sqrt{x_1} + x_2$. Which is true of this consumer's preferences? $(16, 3) \succ (4, 5)$, $(4, 5) \succ (16, 3)$, or $(4, 5) \sim (16, 3)$

Exercise 4.4: Suppose that a consumer's preferences can be represented by the utility function $u(x_1, x_2) = x_1 x_2$. Which is true of this consumer's preferences? $(8, 2) \succ (4, 4)$, $(4, 4) \succ (8, 2)$, or $(4, 4) \sim (8, 2)$

Exercise 4.5: Suppose that a consumer's preferences can be represented by the utility function $u(x_1, x_2) = \sqrt{x_1} + x_2$. What x_2 would the consumer accept (with no x_1) in exchange for the bundle $(9, 4)$. That is, what x_2 solves $(9, 4) \sim (0, x_2)$?

Exercise 4.6: Suppose that a consumer's preferences can be represented by the utility function $u(x_1, x_2) = x_1 x_2$. What bundle with the same amount of both goods would the consumer accept in exchange for the bundle $(9, 4)$. That is, what z solves $(9, 4) \sim (z, z)$?

Exercise 4.7: For the set $X = \{p, q, r\}$, write down a utility function that represents each of these preference relations.

a. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim r, p \succsim r$

b. $p \succsim p, q \succsim q, r \succsim r, p \succsim q, q \succsim p, q \succsim r, r \succsim q, p \succsim r, r \succsim p$

Exercise 4.8: For each of the following utility functions, find the the general MRS and the MRS at the bundle $(2, 2)$.

a. $u(x_1, x_2) = 3x_1 + 2x_2$

b. $u(x_1, x_2) = x_1 x_2$

c. $u(x_1, x_2) = x_1^2 x_2^3$

d. $u(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$

e. $u(x_1, x_2) = 4x_1x_2 + 10$

f. $u(x_1, x_2) = \sqrt{x_1} + x_2$

g. $u(x_1, x_2) = x_1 + x_1x_2$

Exercise 4.9: Of the utility functions above, which represent the same preferences as each other?

Exercise 4.10: Sketch a few indifference curves of the following utility functions.

a. $u(x_1, x_2) = 2x_1 + 3x_2$

b. $u(x_1, x_2) = 2x_1 - 3x_2$

c. $u(x_1, x_2) = min\,\{2x_1, x_2\}$

d. $u(x_1, x_2) = max\,\{x_1, x_2\}$

19.5 Exercises for Chapter 5

Exercise 5.1: Show that the preferences represented by $u(x_1, x_2) = max\{x_1, x_2\}$ is **not convex** by finding two bundles that are indifferent and showing that some convex combination of them is strictly worse than those points.

Exercise 5.2: Are the preferences below monotonic?

a. $u(x_1, x_2) = 2x_1 + 3x_2$

b. $u(x_1, x_2) = 2x_1 - 3x_2$

c. $u(x_1, x_2) = min\,\{2x_1, x_2\}$

d. $u(x_1, x_2) = max\,\{x_1, x_2\}$

Exercise 5.3: For the two bundles $(3, 1)$ and $(1, 3)$ what bundle results from taking a convex combination of these bundles with $t = \frac{1}{2}$?

Exercise 5.4: Show that the bundle from the previous exercise is at least as good as $(3, 1)$ and $(1, 3)$ for the utility function x_1x_2.

19.6 Exercises for Chapter 6

Exercise 6.1: Suppose that prices are $p_1 = 1, p_2 = 2$ and income is $m = 60$. Find the optimal bundle if $u(x_1, x_2) = x_1 + x_2$.

Exercise 6.2: Suppose that prices are $p_1 = 1, p_2 = 2$ and income is $m = 60$. Find the optimal bundle if $u(x_1, x_2) = 2x_1 + 5x_2$.

Exercise 6.3: Suppose that prices are $p_1 = 1, p_2 = 2$ and income is $m = 60$. Find the optimal bundle if $u(x_1, x_2) = x_1 x_2$.

Exercise 6.4: Suppose that prices are $p_1 = 1, p_2 = 2$ and income is $m = 60$. Find the optimal bundle if $u(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$.

Exercise 6.5: Suppose that prices are $p_1 = 1, p_2 = 2$ and income is $m = 60$. Find the optimal bundle if $u(x_1, x_2) = min\{x_1, x_2\}$.

Here are two types of preferences we have not seen so far:

Exercise 6.6: Suppose that prices are $p_1 = 1, p_2 = 2$ and income is $m = 60$. Find the optimal bundle if $u(x_1, x_2) = min\{\frac{1}{3}x_1, x_2\}$.

Exercise 6.7: Suppose that prices are $p_1 = 1, p_2 = 2$ and income is $m = 60$. Find the optimal bundle if $u(x_1, x_2) = max\{x_1, x_2\}$. *Hint, think about what kind of preferences this utility function represents and use your intuition.*

Exercise 6.8: Suppose that prices are $p_1 = 1, p_2 = 2$ and income is $m = 60$. Find the optimal bundle if $u(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$. *Hint, these preferences are convex, you can use the tangency condition.*

Hint, for extra practice, try solving these problems again, but change the prices and income!

19.7 Exercises for Chapter 7

Exercise 7.1: Find the Marshallian demand of x_1 and x_2 for the utility function $U(x_1, x_2) = min\{x_1, \frac{1}{2}x_2\}$.

Exercise 7.2: For $U(x_1, x_2) = min\{x_1, \frac{1}{2}x_2\}$, is x_1:

- Normal or Inferior?
- Ordinary or Giffen?
- Complement, Substitute, or Neither for x_2?

Exercise 7.3: For $U(x_1, x_2) = min\{x_1, \frac{1}{2}x_2\}$, suppose $p_1 = 1, p_2 = 1$, plot the Engle curve for x_1.

Exercise 7.4: For $U(x_1, x_2) = min\{x_1, \frac{1}{2}x_2\}$, suppose $m = 15, p_2 = 1$, find and plot the inverse demand for x_1.

Exercise 7.5: Find the Marshallian demand for the utility function $u(x_1, x_2) = x_1 (x_2)^2$.

Exercise 7.6: Suppose demand for x_1 is $\frac{m-p_2}{p_1}$. Is x_1:

- Normal or Inferior?
- Ordinary or Giffen?
- Complement, Substitute, or Neither for x_2?

Exercise 7.7: Find the Marshallian demand for the utility function $u(x_1, x_2) = \ln(x_1) + x_2$.

Exercise 7.8: Find the Marshallian demand for the utility function $u(x_1, x_2) = x_1 + 2x_2$ by taking the following steps:

- Write down a condition in terms of p_1, p_2, m that determines when consuming only x_1 is optimal.
- Write down a condition in terms of p_1, p_2, m that determines when consuming only x_2 is optimal.
- Write down a condition in terms of p_1, p_2, m that determines when any just-affordable combination of x_1, x_2 is optimal.

Exercise 7.9: When utility is $U(x_1, x_2)$, Marshallian demand is $x_1 = \frac{\frac{1}{2}m}{p_1}$ and $x_2 = \frac{\frac{1}{2}m}{p_2}$. Are the goods:

- Normal or Inferior?
- Ordinary or Giffen?
- Complements, Substitutes, or Neither

19.8 Exercises for Chapter 8

Exercise 8.1:

A consumer has utility function $u(x_1, x_2) = 2x_1 + x_2$.

a. What is the consumer's demand for x_1 and x_2 when $m = 1200$, $p_1 = 100$, and $p_2 = 100$?

b. What is the new demand for x_1 and x_2 with $p_1 = 300$?

c. What income would the consumer need to afford the bundle from part (a) at the new prices in part (b)?

d. What bundle of x_1 and x_2 does the consumer demand when $p_1 = 300$ but with income from part (c)?

e. How much of the consumer's change in demand for x_1 between part (a) and part (b) is due to the substitution effect?

f. How much of the consumer's change in demand for x_1 between part (a) and part (b) is due to the income effect?

Exercise 8.2: A consumer has utility function $u(x_1, x_2) = \min\left\{\frac{1}{2}x_1, x_2\right\}$.

a. What is the consumer's demand for x_1 and x_2 when $m = 1200$, $p_1 = 100$, and $p_2 = 100$?

b. What is the new demand for x_1 and x_2 with $p_1 = 150$?

c. What income would the consumer need to afford the bundle from part (a) at the new prices in part (b)?

d. What bundle of x_1 and x_2 does the consumer demand when $p_1 = 150$ but with income from part (c)?

e. How much of the consumer's change in demand for x_1 between part (a) and part (b) is due to the substitution effect?

f. How much of the consumer's change in demand for x_1 between part (a) and part (B) is due to the income effect?

19.9 Exercises for Chapter 9

Exercise 9.1: A consumer has an endowment of $w_1 = 2$ units of good 1 and $w_2 = 2$ units of good 2. The prices are $p_1 = 4$ and $p_2 = 2$.

a. Write down the consumer's budget equation in terms of their endowment.

b. Sketch the budget line. Make sure to label the slope and intercepts.

c. Suppose the consumer has (gross) demands $x_1 = 1$ and $x_2 = 4$. Are they a net buyer or seller of good 1? What about good 2?

d. What is the **net demand** for each good above?

e. Suppose the price of good 1 increases to $p_2 = 4$. Add this budget line to your graph.

Exercise 9.2: A consumer has an endowment of $w_1 = 10$ units of good 1 and $w_2 = 5$ units of good 2. Prices are initially $p_1 = 2$ and $p_2 = 2$.

a. Write down the consumer's budget equation in terms of their endowment.

b. Sketch the budget line. Make sure to label the slope and intercepts.

c. Suppose the consumer has (gross) demands $x_1 = 12$ and $x_2 = 3$. Are they a net buyer or seller of good 1? What about good 2?

d. What is the **net demand** for each good above?

e. Suppose the price of good 1 decreases to $p_2 = 1$. Is the consumer better off, worse off, or can you not tell?

Exercise 9.3: A consumer has utility $u(x_1, x_2) = x_1 x_2$, an endowment $w_1 = 4$, $w_2 = 8$, and prices $p_1 = 2$, $p_2 = 1$.

a. Find the consumer's (gross) demand for x_1 and x_2.

b. Determine whether the consumer is a net buyer or seller of each good.

Exercise 9.4: A consumer has utility function

$$u(x_1, x_2) = x_1 + \ln(x_2)$$

and endowments $w_1 = 2$, $w_2 = 10$. Prices are $p_1 = 5$ and $p_2 = 1$.

a. Write down the consumer's budget equation.

b. What is the consumer's (gross) demand for x_1 and x_2?

c. Is this consumer a net buyer or seller of x_2? What about x_1?

d. If $p_2 = 1$, what would the price p_1 need to be to make the consumer neither a buyer nor a seller of x_2?

19.10 Exercises for Chapter 10

Exercise 10.1: A consumer has utility function

$$u(c_1, c_2) = c_1 c_2$$

and incomes $m_1 = 1000$ and $m_2 = 1000$ in periods 1 and 2 respectively. The interest rate is $r = 0.25$.

a. Write down the consumer's budget equation.

b. Along the budget line, how much consumption in period 2 would they have to give up to get one more unit of consumption in period 1?

c. What is the optimal amount of c_1 and c_2 for this consumer?

d. How much does the consumer save/borrow in period 1?

e. Suppose the interest rate increases to $r = 0.5$. Is the consumer a saver or borrower? Is the consumer better off?

Exercise 10.2: A consumer chooses consumption in two periods, with period-1 consumption c_1 and period-2 consumption c_2. The consumer's preferences are represented by

$$u(c_1, c_2) = c_1 + c_2.$$

They have incomes $m_1 = 600$ in period 1 and $m_2 = 800$ in period 2, and the market interest rate is $r = 0.2$.

a. Write down the consumer's budget equation.

b. Along the budget line, how much consumption in period 2 would they have to give up to get one more unit of consumption in period 1?

c. Determine the optimal consumption bundle (c_1, c_2).

d. Is the consumer a saver or a borrower in period 1? Explain your answer.

e. Suppose the interest rate increases to $r = 0.3$. Is the consumer now a borrow/saver? Are they better off?

Exercise 10.3: Consider a consumer whose intertemporal preferences are given by

$$u(c_1, c_2) = \min\{c_1, c_2\}.$$

The consumer has income $m_1 = 1000$ in period 1 and $m_2 = 1200$ in period 2, while the interest rate is $r = \frac{2}{3}$.

a. Write down the consumer's budget equation.

b. Along the budget line, how much consumption in period 2 would they have to give up to get one more unit of consumption in period 1?

c. Determine the optimal consumption bundle (c_1, c_2).

d. Is there any interest rate that could make this consumer a saver?

19.11 Exercises for Chapter 11

Exercise 11.1: One hundred consumers have individual demand for some good x of $x = 10 - p$.

a. What is the individual price elasticity of demand for this good? (This will be a function of p).

b. When $p = 9$ what is individual price elasticity of demand?

c. When $p = 9$ what happens to individual demand when price increases by 1%?

d. What is the market demand for this good?

e. What is the market price elasticity of demand for this good? (This will be a function of p).

Exercise 11.2: Ten consumers each have Cobb Douglass utility functions $U(x_1, x_2) = x_1x_2$ they each have income $m_i = 100$

a. Using the tangency condition, what is each person's demand for x_1?

b. What is each individual's income elasticity of demand for x_1?

c. What happens to an individual's demand when their income goes up by 1%?

d. What is each individual's price elasticity of demand for x_1?

e. What happens to an individual's demand when price goes up by 1%? At the individual level, is this good elastic, inelastic, or unit-elastic?

f. What is X_1, the market demand for x_1

g. What is the market price elasticity of demand?

h. What happens to market demand when price goes up by 1%? At the individual level, is this good elastic, inelastic, or unit-elastic?

Exercise 11.3: An individual has demand $x_1 = \frac{m}{p_1+p_2}$ for a good.

a. What is η, their income elasticity of demand for x_1?

b. What is to an individual's demand when their income goes up by 1%?

c. What is $\epsilon_{1,1}$, their price elasticity of demand for x_1?

d. When $p_1 = 1$ and $p_2 = 1$, what is the price elasticity of demand? What happens to their demand when price goes up by 1%? Is this good elastic, inelastic, or unit-elastic?

e. What is $\epsilon_{1,2}$, their *cross* price elasticity of demand for x_1 with respect to price p_2?

f. When $p_1 = 1$ and $p_2 = 1$, what is the cross price elasticity of demand? What happens to their demand for good 1 when price of good 2 goes up by 1%?

Exercise 11.4: Interpret the following elasticities in words:

a. $\epsilon_{1,1} = -10$

b. $\epsilon_{1,1} = -0.1$

c. $\epsilon_{1,2} = 1$

d. $\epsilon_{2,1} = -1$

e. $\eta = 1$

f. $\eta = -\frac{1}{2}$

Exercise 11.5:

Find the price elasticity demand for the following demand functions.

a. $x = \frac{m}{p}$

b. $x = \frac{m-10}{p}$

c. $x = \frac{m}{p^2}$

Exercise 11.6:

Find the income elasticity demand for the following demand functions.

a. $x = \frac{m}{p}$

b. $x = \frac{m-10}{p}$

c. $x = \frac{m}{p^2}$

19.12 Exercises for Chapter 12

Exercise 12.1:

Demand for a good is: $Q_d(p) = 2000 - 30p$. Supply is $Q_s(p) = 10p$.

a. Sketch the supply and demand functions. (Be sure to put p on the y-axis!).

b. What is the equilibrium price and quantity?

c. What is the consumer surplus?

d. What is the producer surplus?

e. If the government imposes a tax of \$20 unit, what will happen to the equilibrium price? Including this tax, how much will consumers pay per unit?

f. On your sketch from part A, label the equilibrium price and quantities before and after the tax is imposed.

g. Label the consumer surplus, producer surplus and area of Dead-Weight-Loss due to the tax.

h. What is the consumer surplus after the tax is imposed?

i. What is the producer surplus after the tax is imposed?

j. What is the government revenue of the tax?

k. What is the dead-weight-loss?

Exercise 12.2: Suppose the demand for a good is given by

$$Q_d(p) = 81 - p,$$

and the supply is given by

$$Q_s(p) = 8p.$$

a. What is the equilibrium price and quantity for the good?

b. What is the elasticity of demand for the good at the equilibrium price?

c. What is the elasticity of supply for the good at the equilibrium price?

d. Is demand elastic, inelastic, or unit elastic?

e. Suppose a tax of $9 per unit is imposed. What is the new equilibrium quantity?

f. Including the tax, how much more do consumers pay per unit compared to the no-tax situation, and how much less do producers receive per unit?

g. What is the dead-weight loss associated with this tax?

Exercise 12.3: Suppose the demand for a good is given by

$$Q_d(p) = 1000 - 8p,$$

and the supply is given by

$$Q_s(p) = 2p.$$

a. Find the equilibrium price and quantity in the market.

b. What is the consumer surplus in equilibrium with no tax?

c. If a tax of $t = 50$ per unit is imposed, what is the new equilibrium quantity?

d. What is the government tax revenue?

e. If a tax of t per unit is imposed, what is the new equilibrium quantity?

f. **(Challenge)** What tax t would maximize government revenue?

Exercise 12.4: Suppose the demand for a good is given by

$$Q(p) = 400 - p,$$

and the supply is given by

$$Q(p) = 3p.$$

a. What is the equilibrium price and quantity in the market?

b. What is the price elasticity of demand at the equilibrium price?

c. Given your answer to part (b), if the price were to increase by 1%, approximately what would happen to the quantity demanded?

d. What is the equilibrium price and quantity if a quantity tax of $t = 200$ is imposed?

e. What is the dead-weight loss associated with this tax?

19.13 Exercises for Chapter 13

Exercise 13.1: For each production function, what are two other bundles on the same isoquant as the bundle $(4, 4)$?

(a) $f(x_1, x_2) = 3x_1 + 2x_2$,

(b) $f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$,

(c) $min\{x_1, x_2\}$

Exercise 13.2: For each of the following production functions, determine the marginal product of x_1 and x_2.

(a) $f(x_1, x_2) = 3x_1 + 2x_2$,

(b) $f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$,

(c) $f(x_1, x_2) = x_1 x_2$

(d) $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{4}}$.

Exercise 13.3: For each of the following production functions, determine whether the production functions has decreasing marginal product for x_1. What about x_2?

(a) $f(x_1, x_2) = 3x_1 + 2x_2$,

(b) $f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$,

(c) $x_1 x_2$

(d) $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{4}}$.

Exercise 13.4: For each of the following production functions, if both inputs are increased by $t > 1$, does output change by more, less, or exactly t? Does this suggest the production function has increasing, decreasing, or constant returns to scale?

(a) $f(x_1, x_2) = 3x_1 + 2x_2$,

(b) $f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$,

(c) $f(x_1, x_2) = x_1 x_2$

(d) $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{4}}$.

Exercise 13.5: For each of the following production functions, derive the technical rate of substitution (TRS) between x_1 and x_2:

(a) $f(x_1, x_2) = 3x_1 + 2x_2$,

(b) $f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$,

(c) $f(x_1, x_2) = x_1 x_2$

(d) $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{4}}$.

19.14 Exercises for Chapter 14

Exercise 14.1: For each of the following production functions, find the conditional factor demands, cost function, and marginal cost. Is marginal cost constant, increasing, or decreasing.

a. $f(x_1, x_2) = x_1 x_2$

b. $f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$

c. $f(x_1, x_2) = min\{x_1, x_2\}$

Exercise 14.2: What is the short-run cost function for producing output y when x_2 is fixed at $x_2 = 1$?

a. $f(x_1, x_2) = x_1 x_2$

b. $f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$

Exercise 14.3: A firm has perfect substitutes production function $f(x_1, x_2) = 2x_1 + 3x_2$. Assume $w_1 = 2, w_2 = 4$.

a. Sketch the firms isoquant for producing output $y = 12$.

b. What is the cost of using only x_1 to produce $y = 12$? What about using only x_2?

c. What are the firm's conditional factor demands for producing output y at these input prices?

d. What is the firm's cost function for producing output y?

Exercise 14.4: Consider a firm with the production function

$$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}},$$

Assume $w_1 = 1$, $w_2 = 1$.

a. Suppose, in the short run, x_2 is fixed at 4, what is the firm's short-run cost function?

b. What is the firm's long-run cost function?

c. Show that for output $y = 4$, the short-run and long-run costs are the same.

d. Show that for output $y = 2$, the short-run cost is greater than the long-run cost.

19.15 Exercises for Chapter 15

Exercise 15.1: Assume a firm is a price-taker. For output price p, set up the profit function for each of the cost functions below. Determine the optimal output y^* as a function of p.

a. $c(y) = y^2$

b. $c(y) = y^2 + 10$

c. $c(y) = y^3$

Exercise 15.2: Suppose a firm has cost function $c(y) = 2y$ and that they are a price-taker with fixed output price p. Set up the profit function. For what range of prices is $y = 0$ optimal? For what range of prices is there no profit maximizing level of output because they can always increase profit by increasing output?

Exercise 15.3: Consider a firm with the production function

$$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}},$$

Assume $w_1 = 4$, $w_2 = 1$, and output price $p = 16$, answer the following:

a. Suppose, in the short run, x_2 is fixed at 4.

b. What is the firm's short-run cost function?

c. Determine the profit-maximizing level of and output y in the short run. How much profit do they make in the short run?

d. What is the firm's long-run cost function?

e. Why is there no long-run profit maximizing level of output?

19.16 Exercises for Chapter 16

Exercise 16.1: Suppose a firm has a cost function $c(y) = y$. Thus, marginal cost is constant at 1. Suppose demand is $y = \frac{100}{p^2}$.

a. What is the inverse demand?

b. What is the monopolist's profit function?

c. Find the optimal output y.

d. What price does the firm charge?

e. What is the market price elasticity of demand at the profit maximizing output?

Exercise 16.2: Suppose demand is $\frac{500}{p-10}$ and a monopolist has cost function $c(y) = y^2$.

a. What is the inverse demand?

b. Set up the firm's profit function.

c. What quantity does the monopolist produce?

d. How much does the monopolist charge?

e. What is its profit?

Exercise 16.3: Suppose demand is $500 - p$ and a monopolist has cost function $c(y) = 100y$.

a. What is the firm's profit function?

b. What is the firm's optimal ouput y?

c. What price do they charge?

d. What is the consumer surplus?

e. What is the producer surplus?

f. What is the dead-weight loss?

Exercise 16.4: Suppose demand is $40 - 2p$ and a monopolist has cost function $c(y) = 5y$.

a. What is the firm's profit function?

b. What is the firm's optimal ouput y?

c. What price do they charge?

d. What is the consumer surplus?

e. What is the producer surplus?

f. What is the dead-weight loss?

19.17 Exercises for Chapter 17

Exercise 17.1: For each of the following real-world examples, identify which type of price discrimination is being used (First-degree, Second-degree, Third-degree, Bundling, or Two-part tariff):

a. A movie theater charging \$12 for adults and \$8 for students

b. A cable company offering a "Sport Package" that includes ESPN, Fox Sports, and NBC Sports for \$30/month but does not offer the channels individually.

c. A newspaper offers a discount to students and educators with a .edu email address.

d. A software company selling a "Professional Suite" that includes word processing software, spreadsheet software, and presentation software, but does not sell the software products individually.

e. A company that sells cybersecurity services to Fortune 100 companies charges different prices to each company, uses what they know about the companies to try and charge each company as close to the maximum that they will pay as possible.

Exercise 17.2: A monopolist sells to two groups of consumers with different demand functions:

- Group A: $y_a = 100 - 2p$
- Group B: $y_b = 80 - p$

The monopolist has cost function $c(y) = 10y$

a. If the monopolist must charge the same price to both groups, what price will they charge?

b. If the monopolist can charge different prices to each group, what price will they charge to Group A?

c. What price will they charge to Group B?

d. How much more profit does the monopolist make by price discriminating?

Exercise 17.3: A monopolist sells to two groups (a and b) of consumers with the following demand functions

$$y_a = 20 - p, \qquad y_b = 12 - P,$$

and has constant marginal (and average) cost of \$4 per unit so their cost function is $c(y) = 4y$.

a. If the monopolist must charge the same price to both groups, what quantity will they choose to sell and what price will they charge?

b. If the monopolist can charge different prices to each group, what price will they charge to Group a?

c. What price will they charge to Group b?

d. How much additional profit does the monopolist earn by using third-degree price-discriminating instead of charging a single price?

Exercise 17.4: A firm sells two products, A and B, to two consumers with the following valuations:

	Product A	Product B	Bundle
Consumer 1	40	60	100
Consumer 2	60	40	100

a. If the firm sells the products separately, what is the maximum profit it can earn?

b. If the firm sells only bundles, what is the maximum profit it can earn?

c. How much more profit does the firm make by bundling?

Exercise 17.5: A firm sells two products, X and Y, to two consumers with the following valuations:

	Product X	Product Y	Bundle
Consumer 1	30	70	100
Consumer 2	70	30	100

a. If the firm sells the products separately, what is the maximum profit it can earn?

b. If the firm sells only bundles, what is the maximum profit it can earn?

c. How much more profit does the firm make by bundling?

Exercise 17.6: A firm has a constant marginal cost of \$2 per unit so their cost function is $c(y) = 2y$. **Each consumer's demand** is $y = 20 - p$.

a. If the firm charges a traditional profit-maximizing per-unit price, what quantity and price maximize the profit from each consumer?

b. How much profit does it make?

c. If the firm uses a two-part tariff, charging $p = 2$ what entry fee will it charge?

d. How much more profit does the firm make using a two-part tariff?

Exercise 17.7: A firm has a constant marginal cost of \$5 per unit so their cost function is $c(y) = 5y$. **Each consumer's demand** is $y = 30 - 2p$.

a. If the firm charges a single price, what price will it charge and how much profit will it make?

b. If the firm uses a two-part tariff, what entry fee will it charge?

c. What per-unit price will it charge?

d. How much more profit does the firm make using a two-part tariff?

19.18 Exercises for Chapter 18

Exercise 18.1: Consider two firms competing in a Cournot model with inverse demand $p(Q) = 50 - Q$ and cost functions $c_1(q_1) = 5q_1$ and $c_2(q_2) = 5q_2$.

a. Write down each firm's profit function.

b. Find each firm's best response function.

c. What is the optimal choice of quantity for firm 1 when $q_2 = 25$?

d. Use symmetry to find the Nash equilibrium quantity for each firm.

e. What is the market price in equilibrium?

f. What is each firm's profit in equilibrium?

Exercise 18.2: Consider $N = 2$ firms competing in a Cournot model with inverse demand $p(Q) = 140 - Q$ and cost functions $c_i(q_i) = 20q_i$.

a. Write down each firm's profit function.

b. Find firm ones best response function.

c. What is the optimal choice of quantity for firm 1 when $q_2 = 20$?

d. Use symmetry to find the Nash equilibrium quantity for each firm.

e. What is the market price in equilibrium?

f. What is each firm's profit in equilibrium?

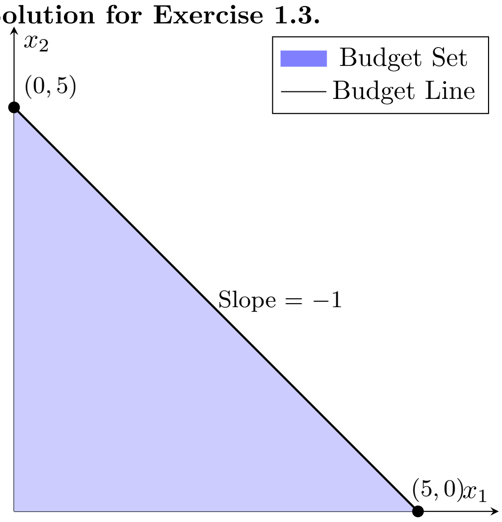
20 Solutions

20.1 Solutions for Chapter 1

Solution for Exercise 1.1.
$x_1 = 3, x_2 = 4$

Solution for Exercise 1.2.
$(0, 0), (2, 3), (4, 1)$

Solution for Exercise 1.3.



Solution for Exercise 1.4.
The consumer will have to give up more x_2 to get extra x_1, thus the budget line will become steeper.

Solution for Exercise 1.5.
Yes, No

Solution for Exercise 1.6.
No, No

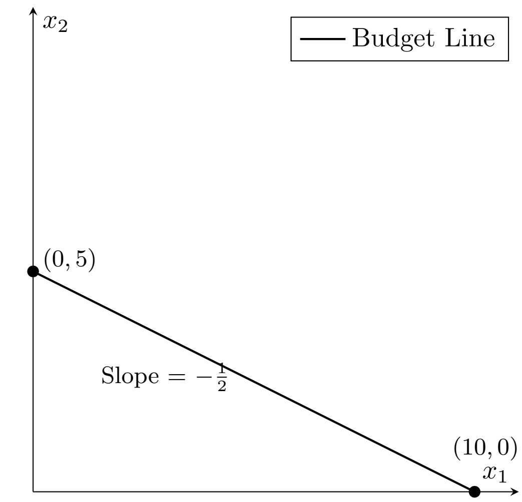
Solution for Exercise 1.7.
Yes, Yes

Solution for Exercise 1.8.
$x_1 + 2x_2 = 10$

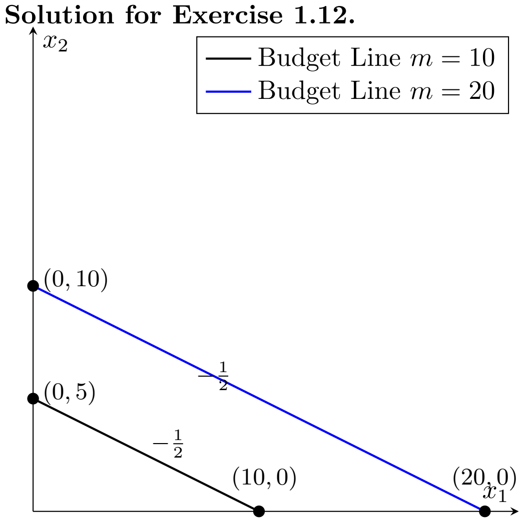
Solution for Exercise 1.9.
10 and 5

Solution for Exercise 1.10.
$-\frac{1}{2}$

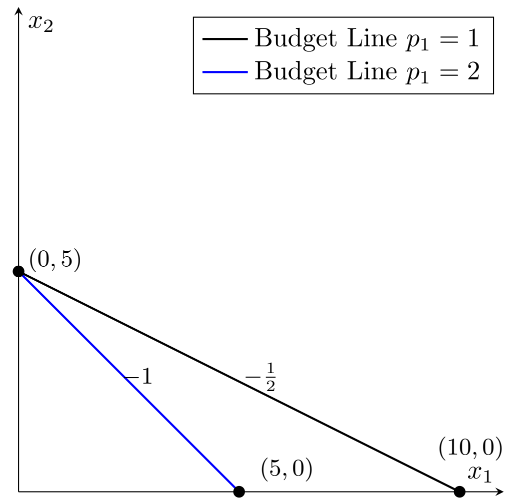
Solution for Exercise 1.11.



Solution for Exercise 1.12.



Solution for Exercise 1.13.



Solution for Exercise 1.14.

$x_1 + (2 + t)x_2 = 10$

20.2 Solutions for Chapter 2

Solution for Exercise 2.1.

Based on the way I normally think about the term "sibling"...

- × Reflexive
- × Complete
- × Transitive
- ✓ Symmetric
- × Asymmetric

Solution for Exercise 2.2.

- ✓ Reflexive
- ✓ Complete
- ✓ Transitive
- × Symmetric
- × Asymmetric

Solution for Exercise 2.3.

- ✓ Reflexive

- × Complete
- ✓ Transitive
- ✓ Symmetric
- × Asymmetric

Solution for Exercise 2.4.

1. Transitive
2. Transitive
3. Not Transitive

Solution for Exercise 2.5.

1. Not Complete (Missing relationship between p, r). Not Transitive (Missing pRr).
2. Complete, Transitive
3. Complete, Transitive
4. Not Complete (Missing relationship between q, r). Not Transitive (Missing qRr).

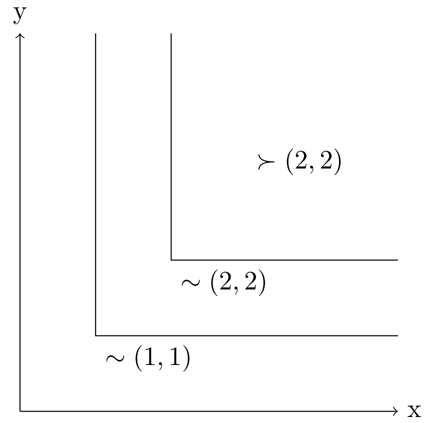
Solution for Exercise 2.6.

1. Reflexive, Transitive, Symmetric
2. Transitive, Asymmetric
3. Reflexive, Complete, Transitive

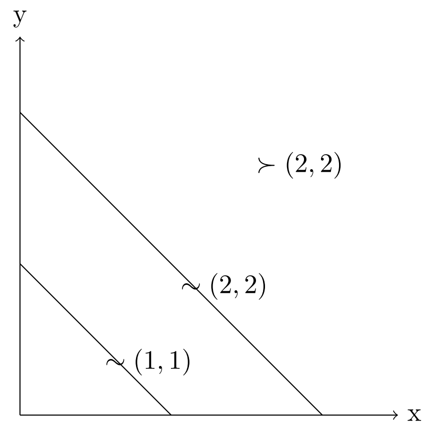
Solution for Exercise 2.7.
A complete relation relates everything to everything else in at least one direction. For a symmetric relation, if a pair are related in one direction, they are related in both directions. Putting these together, a complete and symmetric relation must relate every pair in both directions. Thus, for any set of things, there is only one complete and transitive relation where everything is related to everything else in both directions.

20.3 Solutions for Chapter 3

Solution for Exercise 3.1.



Solution for Exercise 3.2.



Solution for Exercise 3.3.

- $a \sim b \sim c$
- $a \sim b \succ c$
- $a \succ b \sim c \succ d$

Solution for Exercise 3.4.

- $p \succ q, p \succ r, q \succ r$
- None

Solution for Exercise 3.5.

- $p \sim p, q \sim q, r \sim r$

- $p \sim p, q \sim q, r \sim r, p \sim r, p \sim q, q \sim r$

Solution for Exercise 3.6.

- Complete
- Not Transitive

Solution for Exercise 3.7.

- Complete
- Transitive

Solution for Exercise 3.8.

- Not Complete
- Not Transitive

Solution for Exercise 3.9.

- $\{a\}$
- $\{b, c\}$
- $\{c\}$

20.4 Solutions for Chapter 4

Solution for Exercise 4.1.
$b \succ c \succ a$

Solution for Exercise 4.2.
$u(a) = 1, u(b) = 3, u(c) = 2$

Solution for Exercise 4.3.
$(4, 5) \sim (16, 3)$

Solution for Exercise 4.4.
$(4, 4) \sim (8, 2)$

Solution for Exercise 4.5.
$(0, 7)$

Solution for Exercise 4.6.
$(6, 6)$

Solution for Exercise 4.7.

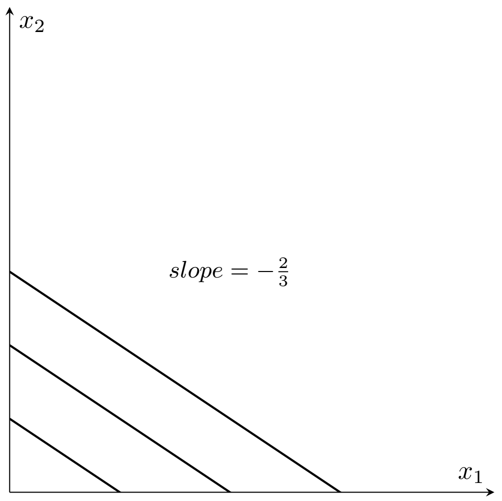
- $u(p) = 3, u(q) = 2, u(r) = 1$
- $u(p) = 3, u(q) = 3, u(r) = 3$

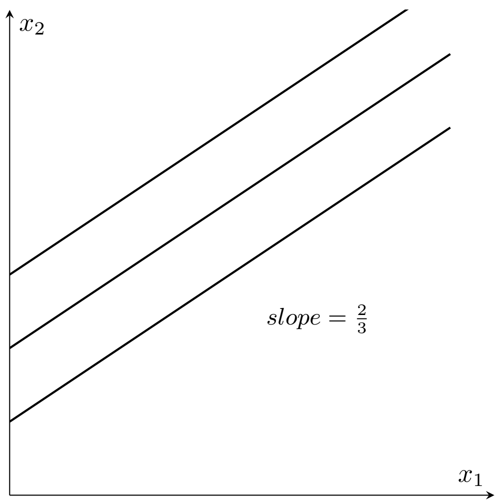
Solution for Exercise 4.8.

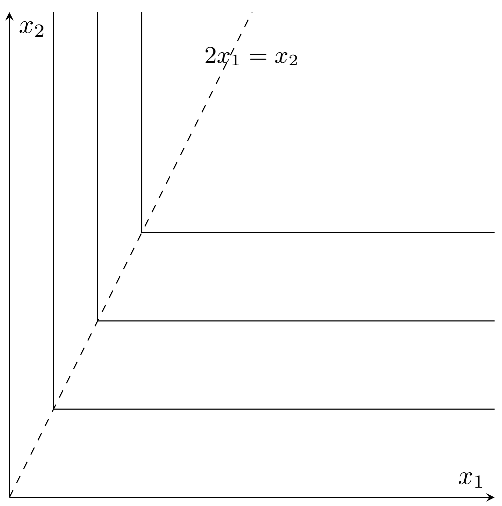
- $-\frac{3}{2}, -\frac{3}{2}$
- $-\frac{x_2}{x_1}, -1$
- $-\frac{2}{3}\frac{x_2}{x_1}, -\frac{2}{3}$
- $-\frac{x_2}{x_1}, -1$
- $-\frac{x_2}{x_1}, -1$
- $-\frac{1}{2\sqrt{x_1}}, -\frac{1}{2\sqrt{2}}$
- $-\frac{x_2+1}{x_1}, -\frac{3}{2}$

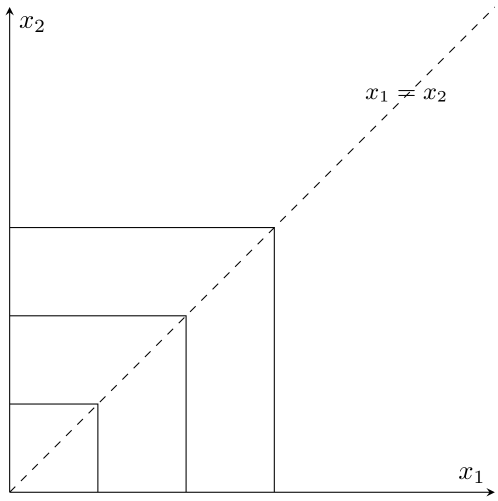
Solution for Exercise 4.9.
b, d, e

Solution for Exercise 4.10.


x_2
$slope = -\frac{2}{3}$
x_1


x_2
$slope = \frac{2}{3}$
x_1


x_2
$2x_1 = x_2$
x_1



20.5 Solutions for Chapter 5

Solution for Exercise 5.1.
$(2,0) \sim (0,2)$ however $(1,1) \succ (2,0)$ and $(1,1) \succ (0,2)$.

Solution for Exercise 5.2.

- Yes.
- No.
- Yes.
- Yes.

Solution for Exercise 5.3.
$(2,2)$

Solution for Exercise 5.4.
$u(2,2) = 4, u(1,3) = u(3,1) = 3$. Thus $(2,2) \succsim (3,1)$ and $(2,2) \succsim (1,3)$.

20.6 Solutions for Chapter 6

Solution for Exercise 6.1.
$(60, 0)$

Solution for Exercise 6.2.
$(0, 30)$

Solution for Exercise 6.3.
$(30, 15)$

Solution for Exercise 6.4.
$(20, 20)$

Solution for Exercise 6.5.
$(20, 20)$

Solution for Exercise 6.6.
$(36, 12)$

Solution for Exercise 6.7.
$(60, 0)$

Solution for Exercise 6.8.
$(40, 10)$

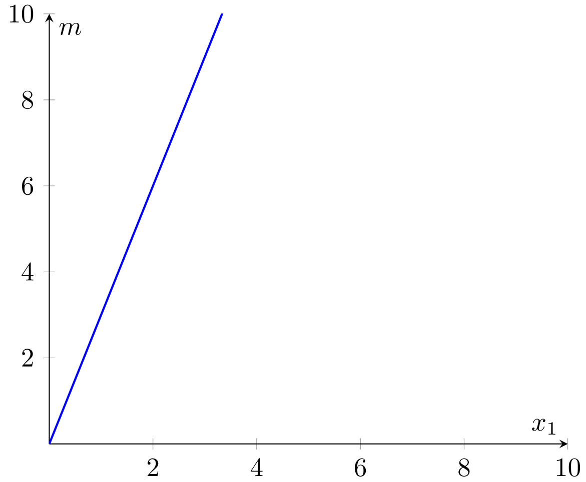
20.7 Solutions for Chapter 7

Solution for Exercise 7.1.
$x_1 = \frac{m}{p_1+2p_2}, x_2 = 2\frac{m}{p_1+2p_2}$

Solution for Exercise 7.2.

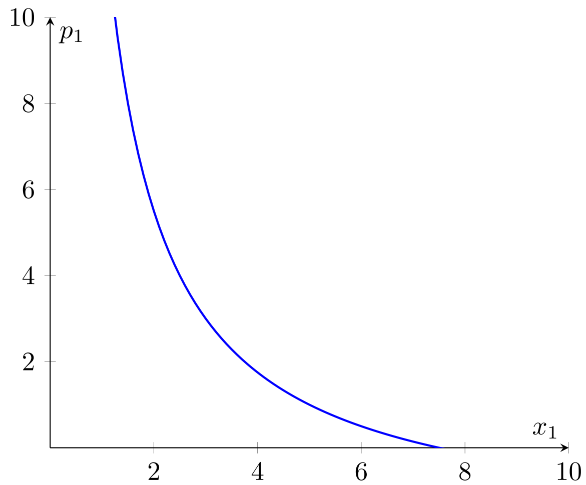
- Normal
- Ordinary
- Complement

Solution for Exercise 7.3.



Solution for Exercise 7.4.

$p_1 = \frac{15}{x_1} - 2$



Solution for Exercise 7.5.

$x_1 = \frac{\frac{1}{3}m}{p_1}, x_2 = \frac{\frac{2}{3}m}{p_2}$

Solution for Exercise 7.6.

- Normal.
- Ordinary.
- Complement.

Solution for Exercise 7.7.

$x_1 = \frac{p_2}{p_1}, x_2 = \frac{m-p_2}{p_2}$

Solution for Exercise 7.8.

- $\frac{m}{p_1} > 2\frac{m}{p_2}$
- $2\frac{m}{p_2} > \frac{m}{p_1}$
- $\frac{m}{p_1} = 2\frac{m}{p_2}$

Solution for Exercise 7.9.

- Normal, Normal.
- Ordinary, Ordinary.
- Neither, Neither.

20.8 Solutions for Chapter 8

Solution for Exercise 8.1.

a. $(12, 0)$

b. $(0, 12)$

c. $\tilde{m} = 3600$

d. $(0, 36)$

e. All 12

f. None

Solution for Exercise 8.2.

a. $(8, 4)$

b. $(6, 3)$

c. $\tilde{m} = 1600$

d. $(8, 4)$

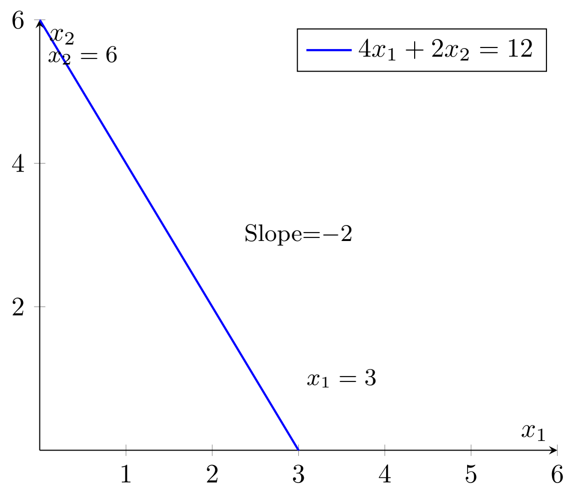
e. None

f. All 2

20.9 Solutions for Chapter 9

Solution for Exercise 9.1.

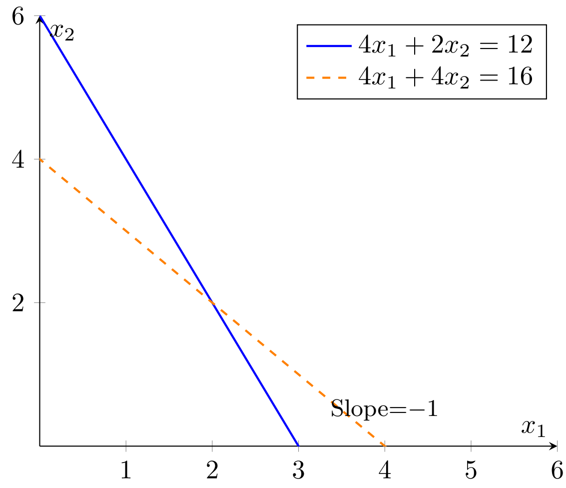
a. $4x_1 + 2x_2 = 12$



b.

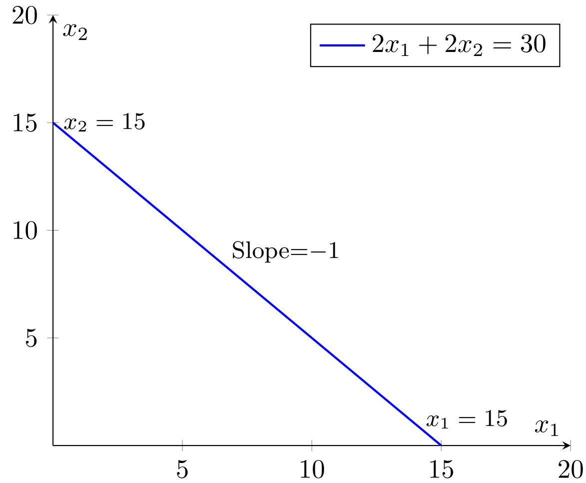
c. Net seller of x_1. Net buyer of x_2.

d. $x_1 - w_1 = -1, x_2 = w_2 = 2$

e. 

Solution for Exercise 9.2.

a. $2x_1 + 2x_2 = 30$

b. 

c. Net buyer of x_1, net seller of x_2.

d. $x_1 - w_1 = 2, x_2 - w_2 = -2$

e. Better off since they were a net buyer of x_1 and its price decreased.

Solution for Exercise 9.3.

a. $x_1 = 4, x_2 = 8$

b. neither, neither

Solution for Exercise 9.4.

a. $5x_1 + x_2 = 20$

b. $x_1 = 3, x_2 = 5$

c. Net Buyer of x_1, Net Seller of x_2

d. $p_1 = 2$

20.10 Solutions for Chapter 10

Solution for Exercise 10.1.

a. $1.25c_1 + c_2 = 2250$

b. 1.25

c. $c_1 = 900, c_2 = 1125$

d. Saves 100

e. Saver, Better Off.

Solution for Exercise 10.2.

a. $1.2c_1 + c_2 = 1520$

b. 1.2

c. $(0, 1520)$

d. Saver since $c_1 < m_1$

e. Still a saver and better off.

Solution for Exercise 10.3.

a. $\frac{5}{3}c_1 + c_2 = \frac{5}{3}1000 + 1200$

b. $\frac{5}{3}$

c. $c_1 = 1075, c_2 = 1075$

d. If the consumer was a saver then $c_1 < m_1$ and $c_2 > m_2$ which implies that $c_1 < 1000$ and $c_2 > 1200$ however, that means that $c_1 \neq c_2$ which is necessary for the bundle to be optimal!

20.11 Solutions for Chapter 11

Solution for Exercise 11.1.

a. $-\frac{p}{10-p}$

b. $-\frac{9}{10-9} = -9$

c. A 1% price increase lowers individual demand by 9%.

d. $X = 100(10-p) = 1000 - 100p$

e. $-\frac{p}{10-p}$

Solution for Exercise 11.2.

a. $x_1 = \dfrac{m}{2p_1}$

b. $\eta = 1$

c. When income increases by 1%, demand increases by 1%.

d. $\epsilon_{1,1} = -1$. This is unit-elastic demand.

e. Demand falls 1% for a 1% price rise (unit-elastic).

f. $X_1 = 10\dfrac{m}{2p_1} = \dfrac{5m}{p_1}$

g. -1

h. A 1% price increases leads to Market demand to decrease by 1%. This is unit-elastic demand.

Solution for Exercise 11.3.

a. $\eta = 1$

b. Demand increases 1% when income increases by 1%.

c. $\epsilon_{1,1} = -\dfrac{p_1}{p_1 + p_2}$

d. With $p_1 = p_2 = 1$, $\epsilon_{1,1} = -\frac{1}{2}$; demand falls 0.5% (inelastic).

e. $\epsilon_{1,2} = -\dfrac{p_2}{p_1 + p_2}$

f. With $p_1 = p_2 = 1$, $\epsilon_{1,2} = -\frac{1}{2}$; a 1% increase in p_2 lowers demand for good 1 by 0.5% (they are complements).

Solution for Exercise 11.4.

a. $\epsilon_{1,1} = -10$: a 1% increase in price decreases demand by 10% (very elastic).

b. $\epsilon_{1,1} = -0.1$: a 1% increase in price decreases demand by 0.1% (very inelastic).

c. $\epsilon_{1,2} = 1$: a 1% increase in p_2 increases demand for good 1 by 1% (substitutes).

d. $\epsilon_{2,1} = -1$: a 1% increase in p_1 decreases demand for good 2 by 1% (complements).

e. $\eta = 1$: a 1% increase in income increases demand by 1%.

f. $\eta = -\frac{1}{2}$: a 1% increase in income decreases demand by 0.5% (inferior good).

Solution for Exercise 11.5.

a. $\epsilon = -1$

b. $\epsilon = -1$

c. $\epsilon = -2$

Solution for Exercise 11.6.

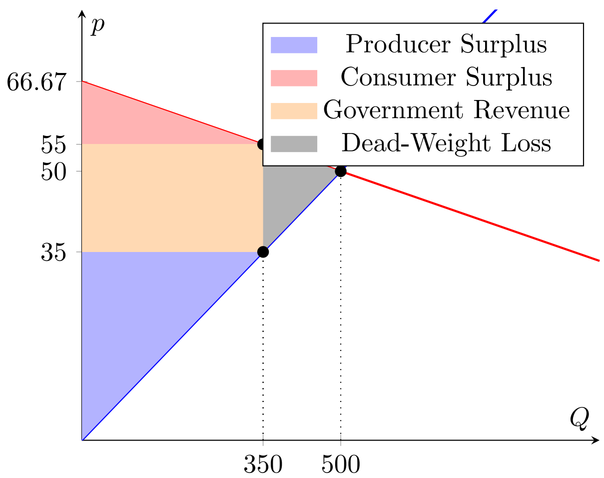
a. $\eta = 1$

b. $\eta = \dfrac{m}{m - 10}$

c. $\eta = 1$

20.12 Solutions for Chapter 12

Solution for Exercise 12.1.

a. 

b. $p^* = 50,\ Q^* = 500$

c. $CS = \dfrac{1}{2}(\frac{2000}{3} - 50) \cdot 500 = \dfrac{12500}{3}$

d. $PS = \dfrac{1}{2} \cdot 50 \cdot 500 = 12500$

e. With tax $t = 20$: $p = 35$ Consumers pay $35 + 20 = 55$.

f. (see sketch)

g. (see sketch)

h. $CS = \dfrac{1}{2}(66.67 - 55) \cdot 350 = \dfrac{6125}{3}$

i. $PS = \dfrac{1}{2} \cdot 35 \cdot 350 = 6125$

j. $G = 20 \cdot 350 = 7000$

k. $DWL = \frac{1}{2} \cdot 20\,(500 - 350) = 1500$

Solution for Exercise 12.2.

a. $p^* = 9,\ Q^* = 72$

b. $\epsilon = \dfrac{\partial\,(81 - p)}{\partial p} \dfrac{p}{81 - p} = -\dfrac{p}{81 - p}$ at $p = 9,\ -\frac{1}{8}$

c. 1

d. Demand is inelastic.

e. With $t = 9$: $p = 8, Q = 64$

f. Consumers pay \$8 more, producers receive \$1 less.

g. $DWL = \frac{1}{2} \cdot 9\,(72 - 64) = 36$

Solution for Exercise 12.3.

a. $p^* = 100,\ Q^* = 200$

b. $CS = \dfrac{1}{2}(125 - 100) \cdot 200 = 2500$

c. With $t = 50$: $p = 60, Q = 120$. Consumers pay $p + 50 = 110$

d. $G = 50 \cdot 120 = 6000$

e. $p^* = \frac{4}{5}(125 - t), Q^* = 200 - \frac{8}{5}t$

f. $G\,(t) = t\left(200 - \frac{8}{5}t\right)$ this is maximized at $t = \frac{125}{2}$ and provides $G = 6250$

Solution for Exercise 12.4.

a. $p^* = 100,\ Q^* = 300$

b. $\frac{\partial(400-p)}{\partial p}\frac{p}{400-p} = -\frac{p}{400-p}$ at $p = 100$ this is $\epsilon = -\frac{1}{3}$

c. A 1% increase in price leads to a $\frac{1}{3}$% decrease in demand.

d. With $t = 200$: $p = 50,\ Q = 150$

e. $DWL = \frac{1}{2} \cdot 200\,(300 - 150) = 15000$

20.13 Solutions for Chapter 13

Solution for Exercise 13.1.

(a) $(0,10),(2,7)$

(b) $(1,9),(0,16)$

(c) $(4,6),(6,4)$

Solution for Exercise 13.2.

(a) $MP_1 = 3,\ MP_2 = 2$

(b) $MP_1 = \dfrac{1}{2\sqrt{x_1}},\ MP_2 = \dfrac{1}{2\sqrt{x_2}}$

(c) $MP_1 = x_2,\ MP_2 = x_1$

(d) $MP_1 = \dfrac{1}{2}x_1^{-1/2}x_2^{1/4},\ MP_2 = \dfrac{1}{4}x_1^{1/2}x_2^{-3/4}$

Solution for Exercise 13.3.

(a) Constant for both inputs

(b) Decreasing for x_1 and x_2

(c) Constant for both inputs

(d) Decreasing for x_1 and x_2

Solution for Exercise 13.4.

(a) Exactly t. Constant returns to scale.

(b) Less than t. Decreasing returns to scale.

(c) More than t. Increasing returns to scale.

(d) Less than t. Decreasing returns to scale.

Solution for Exercise 13.5.

(a) $TRS = -\dfrac{MP_1}{MP_2} = -\dfrac{3}{2}$

(b) $TRS = -\sqrt{\dfrac{x_2}{x_1}}$

(c) $TRS = -\dfrac{x_2}{x_1}$

(d) $TRS = -\dfrac{2x_2}{x_1}$

20.14 Solutions for Chapter 14

Solution for Exercise 14.1.
Conditional factor demands, cost, and marginal cost

a. $f(x_1, x_2) = x_1 x_2$

$$x_1^* = \sqrt{y}\sqrt{\frac{w_2}{w_1}}, \qquad x_2^* = \sqrt{y}\sqrt{\frac{w_1}{w_2}}$$

$$c(y) = 2\sqrt{w_1 w_2}\,\sqrt{y}, \qquad MC = \sqrt{\frac{w_1 w_2}{y}} \text{ (decreasing)}$$

b. $f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$

$$x_1^* = y^{\frac{3}{2}}\sqrt{\frac{w_2}{w_1}}, \qquad x_2^* = y^{\frac{3}{2}}\sqrt{\frac{w_1}{w_2}}$$

$$c(y) = 2\sqrt{w_1 w_2}\, y^{\frac{3}{2}}, \qquad MC = 3\sqrt{w_1 w_2}\, y^{\frac{1}{2}} \text{ (increasing)}$$

c. $f(x_1, x_2) = \min\{x_1, x_2\}$

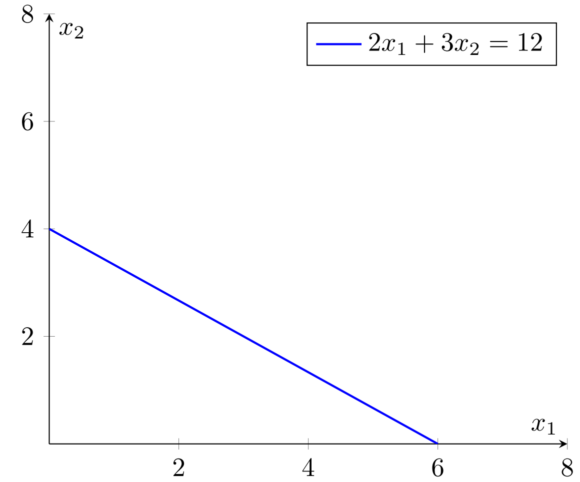
$$x_1^* = y,\ x_2^* = y, \qquad c(y) = (w_1 + w_2)y, \qquad MC = w_1 + w_2 \text{ (constant)}$$

Solution for Exercise 14.2.
Short-run cost with x_2 fixed at 1

a. $w_1 y + w_2$

b. $w_1 y^3 + w_2$

Solution for Exercise 14.3.
$f(x_1, x_2) = 2x_1 + 3x_2,\ w_1 = 2,\ w_2 = 4$



a.

b. Only x_1: $x_1 = 6,$ cost $= 12$; Only x_2: $x_2 = 4,$ cost $= 16$

c. $x_1^* = y/2,\ x_2^* = 0$

d. $c(y) = y,\ \ MC = 1$ (constant)

Solution for Exercise 14.4.
$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}},\ w_1 = w_2 = 1$

a. Short-run cost (with $x_2 = 4$), the required $x_1 = \dfrac{y^2}{4}$. Thus, short run cost of producing y is $\dfrac{y^2}{4} + 4$

b. In the long-run the conditional factor demands are $x_1^* = x_2^* = y$. Thus $c(y) = 2y$.

c. For $y = 4$: $c_{ShortRun} = c_{LongRun} = 8$

d. For $y = 2$: $c_{ShortRun} = 5 > c_{LongRun} = 4$

20.15 Solutions for Chapter 15

Solution for Exercise 15.1.
Price-taking profit maximization

a. $c(y) = y^2$

$$\pi(y) = py - y^2, \quad y^* = \frac{p}{2}, \quad \pi^* = \frac{p^2}{4}$$

b. $c(y) = y^2 + 10$

$$\pi(y) = py - y^2 - 10, \quad y^* = \frac{p}{2}, \quad \pi^* = \frac{p^2}{4} - 10$$

c. $c(y) = y^3$

$$\pi(y) = py - y^3, \quad y^* = \sqrt{\frac{p}{3}}, \quad \pi^* = p\left(\frac{p}{3}\right)^{\frac{1}{2}} - \left(\frac{p}{3}\right)^{\frac{3}{2}} = \frac{2\,p^{3/2}}{3\sqrt{3}}$$

Solution for Exercise 15.2.
Linear cost $c(y) = 2y$

$$\pi(y) = py - 2y = (p-2)y$$

- If $p < 2$: $y^* = 0$
- If $p = 2$: any $y \geq 0$ yields zero profit (take $y^* = 0$)
- If $p > 2$: profit increases without bound as $y \to \infty$ (no finite maximizer)

Solution for Exercise 15.3.
Cobb–Douglas technology $f(x_1, x_2) = x_1^{1/2} x_2^{1/2}$**,** $w_1 = 4$**,** $w_2 = 1$**,** $p = 16$

(b) Short-run cost with $x_2 = 4$ fixed:

$$x_1 = \frac{y^2}{4}, \qquad c_{SR}(y) = 4x_1 + 4 = y^2 + 4$$

(c) $\pi_{SR}(y) = 16y - y^2 - 4,\ y^* = 8,\ \pi^* = 60$

(d) Long-run conditional demands

$$x_1^* = \tfrac{1}{2}y, \ \ x_2^* = 2y$$

Long-run cost

$$c_{LR}(y) = 4y$$

(e) With $p = 16 > MC_{LR} = 4$, profit $\pi_{LR} = 12y$ rises without bound, so no finite long-run optimizer.

20.16 Solutions for Chapter 16

Solution for Exercise 16.1.

a. $p = \frac{10}{y^{1/2}}$

b. $\pi(y) = y\left(\frac{10}{y^{1/2}}\right) - y$

c. $y^* = 25$

d. $p^* = 2$

e. $\epsilon = -2$

Solution for Exercise 16.2.

a. $p = 10 + \dfrac{500}{y}$

b. $\pi(y) = y\left(10 + \dfrac{500}{y}\right) - y^2$

c. $y^* = 5$

d. $p^* = 110$

e. $\pi^* = 525$

Solution for Exercise 16.3.

a. $\pi(y) = y(500 - y) - 100y$

b. $y^* = 200$

c. $p^* = 300$

d. $CS = \frac{1}{2}(500 - 300) \cdot 200 = 20000$

e. $PS = \pi^* = 40000$

f. $DWL = \frac{1}{2}(300 - 100)(400 - 200) = 20000$

Solution for Exercise 16.4.
Demand $y = 40 - 2p,\ c(y) = 5y$

a. $\pi(y) = y(20 - \frac{y}{2}) - 5y$

b. $y^* = 15$

c. $p^* = 12.5$

d. $CS = \frac{1}{2}(20 - 12.5) \cdot 15 = 56.25$

e. $PS = 112.5$

f. $DWL = \frac{1}{2}(12.5 - 5)(30 - 15) = 56.25$

20.17 Solutions for Chapter 17

Solution for Exercise 17.1.

a. Third-degree

b. Bundling

c. Third-degree

d. Bundling

e. First-degree

Solution for Exercise 17.2.

a. $p^* = 35$

b. $p_A^* = 30$

c. $p_B^* = 45$

d. Gain $= 150$

Solution for Exercise 17.3.

a. $Y^* = 12,\ p^* = 10$

b. $p_a^* = 12$

c. $p_b^* = 8$

d. 8

Solution for Exercise 17.4.
Valuations Consumer 1: $(40, 60, 100)$; Consumer 2: $(60, 40, 100)$

a. $p_A = 40,\ p_B = 40,\ \pi = 160$

b. Bundle $p = 100,\ \ \pi = 200$

c. Bundling adds 40

Solution for Exercise 17.5.
Valuations Consumer 1: $(30, 70, 100)$; Consumer 2: $(70, 30, 100)$

a. $p_X = 70,\ p_Y = 70,\ \pi = 140$

b. Bundle $p = 100,\ \pi = 200$

c. Bundling adds 60

Solution for Exercise 17.6.
Two-part tariff, $c(y) = 2y,\ y = 20 - p$

a. $y^* = 9,\ p^* = 11$

b. $\pi = 81$

c. Entry fee $T = 162$

d. $\pi = 162,$ gain 81

Solution for Exercise 17.7.
$y = 30 - 2p,\ c(y) = 5y$

a. $y^* = 10,\ p^* = 10,\ \pi = 50$

b. Consumer surplus at $p = 5$: $CS = 100$ (this is also the entry fee)

c. Unit price under two-part tariff $p^* = 5$

d. 50

20.18 Solutions for Chapter 18

Solution for Exercise 18.1.

a. $\pi_1 = q_1(50 - q_1 - q_2) - 5q_1$
$\pi_2 = q_2(50 - q_1 - q_2) - 5q_2$

b. $BestResponse_1 :\ q_1 = \dfrac{45 - q_2}{2}\ BestResponse_2 :\ q_2 = \dfrac{45 - q_1}{2}$

c. $q_1 = 10$

d. $q_1 = q_2 = 15$

e. $p = 20$

f. $\pi_1 = \pi_2 = 225$

Solution for Exercise 18.2.

a. $\pi_1 = q_1(140 - (q_1 + q_2)) - 20q_1$
$\pi_2 = q_2(140 - (q_1 + q_2)) - 20q_2$

b. $q_1 = \frac{120 - q_2}{2}$

c. $q_1 = 50$

d. $q_1 = q_2 = 40$

e. $p = 60$

f. $\pi_i = 1600$

Part II

Appendix

A Calculus

A.1 Power Rule

Info A.1: Power Rule. The **power rule** states that if $f(x) = x^n$, then the derivative $f'(x) = \frac{\partial f(x)}{\partial x} = nx^{n-1}$.[a]

[a] You may be used to seeing the notation $\frac{df(x)}{dx}$ instead of $\frac{\partial f(x)}{\partial x}$ when taking the derivative of a function with only one variable, but throughout this text I used ∂ everywhere to denote a derivative (when there is only one variable) and a partial derivative (when there are multiple variables).

For example, if $f(x) = x^\alpha$ then $f'(x) = \frac{\partial f(x)}{\partial x} = \alpha x^{\alpha-1}$.

A.2 Derivative of Natural Logarithm

Info A.2: Derivative of Natural Log. If $f(x) = \ln(x)$, then the derivative is $f'(x) = \frac{\partial f(x)}{\partial x} = \frac{1}{x}$.

A.3 Sum Rule

Info A.3: Sum Rule. The **sum rule** states that if you have two functions $u(x)$ and $v(x)$, then the derivative of their sum $f(x) = u(x) + v(x)$ is given by:

$$\frac{\partial f(x)}{\partial x} = \frac{\partial u(x)}{\partial x} + \frac{\partial v(x)}{\partial x}$$

For example, if $f(x) = x^2 + \ln(x)$, then the derivative is given by:

$$\frac{\partial f(x)}{\partial x} = 2x + \frac{1}{x}$$

A.4 Product Rule

Info A.4: Product Rule. The **product rule** is a formula used to find the derivative of the product of two functions. If you have two functions $u(x)$ and $v(x)$, then the derivative of their product $y = u(x)v(x)$ is given by:

$$\frac{\partial y}{\partial x} = u(x) \cdot \frac{\partial v}{\partial x} + v(x) \cdot \frac{\partial u}{\partial x}$$

For example, if $f(x) = x^2 \cdot \ln(x)$, then the derivative is given by:

$$\frac{\partial f(x)}{\partial x} = x^2 \cdot \frac{1}{x} + \ln(x) \cdot 2x = x + 2x\ln(x)$$

A.5 Chain Rule

Info A.5: Chain Rule. The **a** states that if you have a composite function $y = g(f(x))$, then the derivative $\frac{\partial y}{\partial x}$ is given by $\frac{\partial y}{\partial f} \cdot \frac{\partial f}{\partial x}$.

For example, if $f(x) = ln(x^2)$ then $f'(x) = \frac{\partial f(x)}{\partial x} = \left(\frac{1}{x^2}\right) 2x = \frac{2}{x}$.

A.6 Partial Derivatives

When dealing with functions of more than one variable, partial derivatives provide a way to explore how changes in one input variable impact the function, keeping others fixed.

Info A.6: Partial Derivatives. A **partial derivative** of a function of multiple variables is its derivative with respect to one of those variables, with the other variables held constant (imagine it is some fixed number). If you have a function $f(x, y)$, then the partial derivative of f with respect to x is denoted as $\frac{\partial z}{\partial x}$, and with respect to y is $\frac{\partial z}{\partial y}$.

For example, if $f(x, y) = x^2y + y^3$, the partial derivative of f with respect to x is $\frac{\partial f}{\partial x} = 2xy$, and with respect to y is $\frac{\partial f}{\partial y} = x^2 + 3y^2$.

A.7 Second Derivative

The **second derivative** provides information about the curvature of a function. It measures how the first derivative (the slope) changes as a variable changes.

Info A.7: Second Derivative. For a function $f(x)$ with a first derivative $f'(x)$, then the second derivative $f''(x) = \frac{\partial^2 f(x)}{\partial x^2}$ is simply the of the $f'(x)$.

For example, if $f(x) = x^3$, then the first derivative is $f'(x) = 3x^2$ and the second derivative is $f''(x) = 6x$.

A.8 Exercises for Appendix Chapter A

Exercise A.1: Find the derivative of $f(x) = x^5$.

Exercise A.2: Find the derivative of $f(x) = 2x^7 + 3x^4$ using the power rule.

Exercise A.3: Find the derivative of $f(x) = \frac{10}{x^4}$.

Exercise A.4: Find the derivative of $f(x) = \ln(x^3 + 1)$.

Exercise A.5: Find the derivative of $f(x) = \sqrt{\ln(x)}$.

Exercise A.6: Find the partial derivatives of $f(x, y) = x^3+y^3+3xy$ with respect to both x and y.

Exercise A.7: Find the partial derivatives of $f(x, y) = xy$ with respect to both x and y.

Exercise A.8: Find the partial derivatives of $f(x, y) = x^2y^2$ with respect to both x and y.

Exercise A.9: Find the partial derivatives of $f(x, y) = x^\alpha y^\beta$ with respect to both x and y.

Exercise A.10: Find the partial derivatives of $f(x, y) = x^2y^3+x^3y$ with respect to both x and y.

Exercise A.11: Find the derivative of $f(x) = \frac{x^5}{x^2+1}$.

Exercise A.12: Find the partial derivatives of $f(x, y) = x\ln(y) + y\ln(x)$ with respect to both x and y.

Exercise A.13: Find the second derivative of $f(x) = x^6 + 5x^2$.

Exercise A.14: Find the second derivative of $f(x) = ln(x)$.

A.9 Solution for Appendix Chapter A

Exercise A.1: $f'(x) = 5x^4$.

Exercise A.2: $f'(x) = 14x^6 + 12x^3$

Exercise A.3: $f'(x) = \frac{-40}{x^5}$.

Exercise A.4: $f'(x) = \frac{3x^2}{x^3+1}$.

Exercise A.5: $f(x) = \frac{1}{2x\sqrt{\ln(x)}}$.

Exercise A.6: $\frac{\partial f(x,y)}{\partial x} = 3x^2 + 3y$
$\frac{\partial f(x,y)}{\partial y} = 3y^2 + 3x$

Exercise A.7: $\frac{\partial f(x,y)}{\partial x} = y$
$\frac{\partial f(x,y)}{\partial y} = x$

Exercise A.8: $\frac{\partial f(x,y)}{\partial x} = 2xy^2$
$\frac{\partial f(x,y)}{\partial x} = 2x^2y$

Exercise A.9: $\frac{\partial f(x,y)}{\partial x} = \alpha x^{\alpha-1}y^{\alpha}$
$\frac{\partial f(x,y)}{\partial x} = \alpha x^{\alpha}y^{\alpha-1}$

Exercise A.10: $\frac{\partial f(x,y)}{\partial x} = 2xy^3 + 3x^2y$
$\frac{\partial f(x,y)}{\partial x} = 3x^2y^2 + x^3$

Exercise A.11: $f'(x) = \frac{5x^4}{x^2+1} - \frac{2x^6}{(x^2+1)^2}$.

Exercise A.12: $\frac{\partial f(x,y)}{\partial x} = \ln(y) + \frac{y}{x}$
$\frac{\partial f(x,y)}{\partial x} = \frac{x}{y} + \ln(x)$

Exercise A.13: $f''(x) = 30x^4 + 10$.

Exercise A.14: $f''(x) = -\frac{1}{x^2}$.

B Unconstrainted Optimization

B.1 Unconstrained One-Dimensional Optimization

Optimization involves finding the minimum or maximum of a function $f(x)$. Here, we focus on instances where f is one-dimensional. The goal is to determine the value of x that maximizes (or minimizes) $f(x)$. *Unconstrained* means that we will not place any restrictions on what x can be.

Imagine that you are hiking on a mountain trail. If the slope of the trail is positive, then moving forward will bring you to a higher point. If the slope of the trail is negative, then moving *backward* will bring you to a higher point. Thus, **the slope must be zero at the peak**. This is demonstrated in Figure Figure B.1.

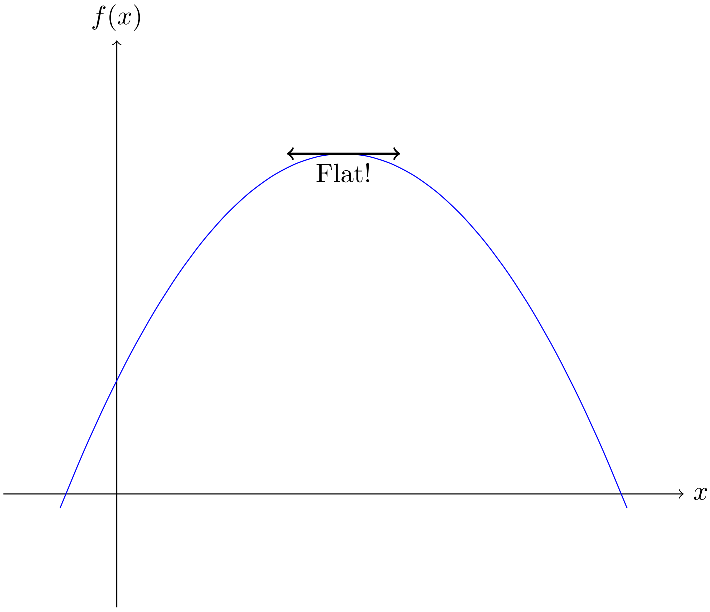


Figure B.1: Slope is Zero at the Peak

One issue with using this fact to find a maximum is that the slope can also be zero at a minimum and also at places that are "local" maxima.

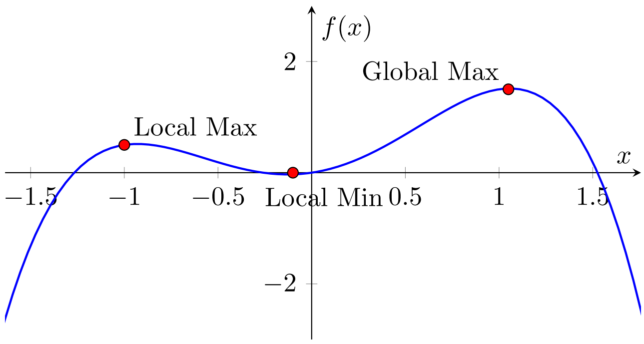


Figure B.2: Not every point of zero slope is a global maximum!

To account for this, we should remember that when we find a point of zero slope, it is only a *candidate* for a global maximum.

Info B.1: Unconstrained Optimization. How to find the unconstrained maximum of a one-dimensional function: For a function $f(x)$:

1. Find the first derivative $f'(x)$.
2. Set the first derivative to zero: $f'(x) = 0$.
3. Solve for x. These are your candidates.
4. Which, if any is a global max?

B.2 Unconstrained Multi-Dimensional Optimization

The intuition of the slope being zero at the maximum holds even when there are multiple directions in which you can move. Imagine trying to find the peak of a mountain when you are not on a trail. You can move east/west or north/south. In fact, you can also move in combinations of these directions, like the northwest. But at the peak, you better not be able to move east-/west and get to a higher altitude. **The slope has to be zero in the east/west direction**. Similarly, **the slope has to be zero in the north/south direction.** One of the nice things about *smooth* functions is that if the slope is zero in these two cardinal directions, it will be zero even if you try to move northwest, or southeast, or any other direction. Figure 6.1 demonstrates this. Notice that at the peak, the slope is zero in both the x direction and the y direction, and also in all other directions.

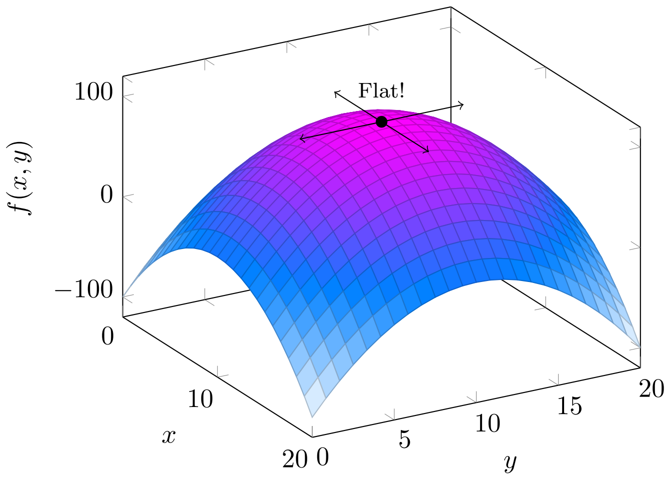


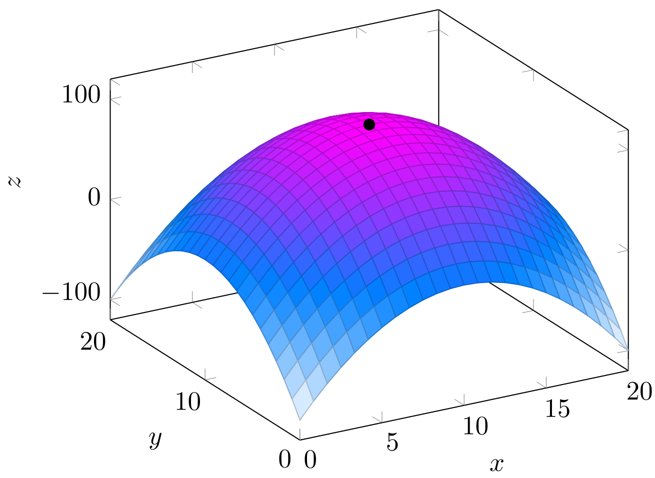
Figure B.3: Slope is Zero in All Directions!

Info B.2: Unconstrained Multi-Dimensional Optimization. To maximize a function $f(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$:

1. Find all partial derivatives $\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$.
2. Set all partial derivatives to zero: $\frac{\partial f}{\partial x_i} = 0$.
3. Solve the resulting system of equations for $(x_1, ..., x_n)$. These solutions are your candidates.
4. Determine which, if any, of these solutions is a global maximum.

Let's try maximizing $100 - (x - 10)^2 - (y - 10)^2$.

Let's look at this function first. The global maximum (black dot) occurs where $x = 10$ and $y = 10$.



Now, we confirm that this is the maximum formally using the procedure in B.2. The partial derivatives are $\frac{f(x,y)}{x} = -2(x - 10)$ and $\frac{f(x,y)}{y} = -2(y - 10)$.
Setting these to zero, we get the equations:

$$\frac{f(x, y)}{x} = -2(x - 10) = 0$$
$$\frac{f(x, y)}{y} = -2(y - 10) = 0$$

Solving these gives us the (x, y) where the function has zero slope. The only solution is $x = 10$, $y = 10$.
We can see by inspecting the function that this must be the global maximum.

B.3 Exercises

Assume for each of the following problems that $x \geq 0$ and $y \geq 0$.

Exercise B.1: Maximize the function $f(x) = -x^2 + 4x + 4$.

Exercise B.2: Maximize the function $f(x) = ln(x) - \frac{1}{4}x + 4$.

Exercise B.3: Maximize the function $f(x, y) = -x^2 - y^2 + 2x + 2y$.

C Some Old Exam Questions

Caveat

These questions were taken from my old exams from similar courses. Please note that these questions are not necessarily exhaustive of the material that might be covered on an exam. Please also refer to the **key topics** of each chapter and the **exercises for each chapter**.

Question C.1: Fill in the blanks:

a. An ordinary good is one for which demand ___________ when ___________ increases.

b. If $(4, 0) \succ (2, 2)$ and $(0, 4) \succ (2, 2)$, then preferences are not ___________.

Question C.2: Fill in the blanks:

a. If a consumer is a borrower and the interest rate ___________, they will always remain a borrower.

b. A good is inferior. If ___________ decreases then demand will ___________.

c. The ___________ measures the slope of indifference curves.

Question C.3: Fill in the blanks:

a. If a consumer is a net buyer of some good, then their gross demand is ___________ than their endowment for that good.

b. If an increase in income causes demand for a good to fall, the good is called ___________.

c. A consumer who prefers $(1, 2)$ to $(2, 3)$ has preferences that are not ___________.

Question C.4: Fill in the blanks:

a. A normal good is one for which demand ___________ as income increases.

b. A person is a net borrower. If the interest rate ___________, they will remain a net borrower and be strictly better off.

c. A consumer who strictly prefers both bundles $(2, 0)$ and $(0, 2)$ to $(1, 1)$ has preferences that are not ___________.

Question C.5: Fill in the blanks:

a. An inferior good is one where demand ___________ when ___________ increases.

b. If a consumer can compare every pair of bundles and state a preference or indifference, their preferences are ___________.

Question C.6: Fill in the blanks:

a. A Giffen good is one for which demand ___________ as its own price increases.

b. When someone prefers to have less of a good, their preferences are not ___________.

Question C.7: Consider the following preference relation on the set $\{a, b, c\}$:

$$a \succsim a, b \succsim b, c \succsim c, a \succsim b, b \succsim a, a \succsim c, b \succsim c$$

a. Is it complete?

b. Write the indifference relation.

c. Write the strict preference relation.

d. Write the preferences in chain notation.

e. Write a utility function that represents these preferences.

Question C.8: Consider the following preference relation on the set $\{a, b, c\}$:

$$a \succsim a, b \succsim b, c \succsim c, a \succsim b, b \succsim c, a \succsim c$$

a. Is it complete?

b. Write the strict preference relation $\succ$.

c. Write the indifference relation $\sim$

d. Write the preferences in chain notation.

e. Write any utility function that represents these preferences.

Question C.9:

A person's utility is $U(x_1, x_2) = x_1 x_2$. Prices are p_1, p_2 and they have income m.

a. Write down the equation for their budget line.

b. What is the slope of the budget line?

c. What is the slope of their indifference curve at the point $(3, 2)$.

d. Write down an equation that implies that the slope of the budget equation is the same as their indifference curve at the point (x_1, x_2). (Hint: This is the "Tangency" condition.)

e. What is their Marshallian demand for x_1 and x_2?

f. If $p_1 = 2$ and $p_2 = 1$, and $m = 20$ what bundle is optimal?

g. As income goes up, what happens to their demand for x_1? Is x_1 a normal good or inferior good?

h. As the price p_2 goes up, what happens to demand for x_1? Is x_1 a compliment, substitute or neither for x_2?

Question C.10: Suppose someone's utility is $u(x_1, x_2) = 5ln(x_1) + x_2$. Prices are p_1 and p_2.

a. What is the slope of this person's indifference curve at the point (x_1, x_2).

b. What is the slope of the budget line?

c. What is the consumer's demand for x_1 and x_2 when $p_1 = 1$, $p_2 = 1$ and $m = 10$?

d. *More Challenging:* Suppose $p_2 = 2$ and we observe they buy $x_1 = 1$ and spend the rest of their money on x_2. Determine what p_1 must be.

Question C.11: A consumer's demands are $x_1 = \frac{1}{2}\frac{m}{p_1}$ $x_2 = \frac{1}{2}\frac{m}{p_2}$ and their income is m. Price are p_1 and p_2, respectively.

a. What is their budget equation when $m = 1200$, $p_1 = 100$, and $p_2 = 100$?

b. How much x_1, x_2 do they demand when $m = 1200$, $p_1 = 100$, and $p_2 = 100$?

c. The price of x_1 increases to $p_1 = 300$ what is their change in demand for x_1 (the total effect)?

d. How much money would they need to afford the old bundle at the new prices? Call this $\tilde{m}$.

e. What bundle would they choose with the prices $p_1 = 300, p_2 = 100$ and income $\tilde{m}$?

f. Of the total change in demand due to the substitution effect and how much is due to the income effect?

Question C.12: A consumer will earn $m_1 = 200$ this month and $m = 600$ next month. Let c_1 and c_2 be their consumption in months 1 and 2.

a. Suppose the interest rate is $r = \frac{1}{2}$. How much can they consume in month 1 if they consume nothing in month 2? How much can they consume in month 2 if they consume nothing in month 1?

b. Sketch their intertemporal budget line (label intercepts and slope).

c. If their utility is $U(c_1, c_2) = c_1c_2$, what is their optimal bundle of (c_1, c_2)?

d. Are they a borrower or a saver?

e. If the interest rate decreases, are they better-off, worse-off, or can you not tell?

Question C.13: A consumer has utility

$$U(x_1, x_2) = \min\left\{x_1, \frac{1}{4}x_2\right\},$$

a. Write their budget equation.

b. Derive the Marshallian demand for x_1. That is, what is their optimal choice of x_1 as a function of p_1, p_2, m

c. Sketch the Engel curve for x_1 when $p_1 = 4$, $p_2 = 1$.

Question C.14: A consumer's utility for consumption today (c_1) and next year (c_2) is

$$u(c_1, c_2) = \min\{c_1, c_2\},$$

They will receive incomes $m_1 = 600$, $m_2 = 1500$, and the interest rate is r.

a. Write the intertemporal budget equation.

b. What is the optimal bundle of (c_1, c_2) for them when $r = 0.25$?

c. Is the consumer a borrower or saver when $r = 0.25$?

d. If r decreases to 0.1, is the consumer a borrower or saver?

Question C.15: A consumer has utility

$$U(x_1, x_2) = x_1 + x_2,$$

with $p_1 = 4$, $p_2 = 2$, and $m = 20$.

a. Write the budget constraint.

b. What is the slope of the indifference curves?

c. What is the optimal bundle for the consumer?

d. If $p_2 = 5$ what is the optimal bundle?

e. How much income $\tilde{m}$ would the consumer need to buy the old bundle (from when $p_2 = 2$) at the new prices?

f. How much of the change from (b) to (c) is due to the substitution effect?

Question C.16: A consumer has utility

$$U(x_1, x_2) = x_1 x_2,$$

an endowment of $(\omega_1, \omega_2) = (10, 0)$, and faces prices $p_1 = p_2 = 1$.

a. Write the budget equation.

b. What is the marginal rate of substitution?

c. What is the demand for x_1?

d. Is the consumer a net buyer or seller of x_1?

Question C.17: A consumer has

$$u(x_1, x_2) = \min\left\{\frac{1}{3}x_1, x_2\right\},$$

Suppose $p_1 = 1$, $p_2 = 6$, and $m = 900$.

a. Write the equation for the budget line.

b. What is the consumer's demand for x_2?

c. If p_2 changes to 12, what is the total effect of the change in demand for x_2?

d. how much of the change in demand for x_2 is due to the substitution effect? How much is due to the income effect?

Question C.18: A consumer with endowment $(\omega_1, \omega_2) = (5, 5)$ has utility

$$u(x_1, x_2) = x_1 + x_2,$$

with $p_1 = 2$ and $p_2 = 4$.

a. Write the budget equation.

b. What is the optimal bundle?

c. In (b), is the consumer a net buyer, or net seller of x_1?

d. If p_1 decreases to 1, is the consumer a buyer or seller of x_1?

Question C.19: A consumer has utility

$$U(x_1, x_2) = \min\{x_1, 2x_2\},$$

with income m and prices p_1, p_2.

a. Write the budget line.

b. What is the "no waste condition" for this consumer?

c. Derive the Marshallian demands for x_1 and x_2.

Concept Check: Fill-in-the-Blank

Question C.20
Fill in all blanks:

a. For a firm, profit maximization implies ___________ minimization.

b. If a firm doubles its inputs but output increases by less than double, its production function has ___________ returns to scale.

c. A monopolist can never be profit-maximizing if demand is ___________.

Equilibrium Problems

Question C.21
Suppose the demand for a good is given by $Q_d = 200 - 40p$ and market supply is $Q_s = 10p$.

a. What is the equilibrium price and quantity in the market?

b. What is the price elasticity of demand at the equilibrium price?

c. What is the equilibrium price and quantity with a quantity tax of $t = \frac{5}{2}$?

d. What is the dead-weight loss associated with this tax?

Question C.22
Suppose market demand is $Q_d = 300 - 2p$ and market supply is $Q_s = p$

a. What is the equilibrium price and quantity in this market?

b. What is the price elasticity of demand at the equilibrium price?

c. If the price rises by 1%, approximately how does quantity demanded change?

d. What are the equilibrium price and quantity if a quantity tax $t = 75$ is imposed?

e. What is the dead-weight loss from this tax?

Question C.23
Demand and supply for a good are $Q_d = 81 - p$ and $Q_s = 8p$.

a. Find the equilibrium price and quantity.

b. Compute the elasticity of demand at the equilibrium price. Is demand elastic, inelastic, or unit-elastic?

c. If a $t = 9$ per-unit tax is imposed, what is the new equilibrium quantity?

d. How much more do consumers pay per unit and how much less do producers receive?

e. Calculate the associated dead-weight loss.

Question C.24
Demand is $Q_d = 1000 - 8p$ and supply is $Q_s = 2p$.

a. Find the equilibrium price and quantity.

b. What is consumer surplus in equilibrium?

c. If the government levies a \$50 tax, what is the new equilibrium quantity?

d. Compute the dead-weight loss.

e. *Challenge:* Which tax rate maximizes government revenue?

Cost-Minimization Problems

Question C.25
A firm produces output y with $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$.

a. Suppose $w_1 = 1$ and $w_2 = 1$. Find the conditional demands for x_1, x_2.

b. Find the cost function.

c. If x_2 is fixed in the short run at $x_2 = 1$, what is the short-run cost of producing y?

d. If the output price is $p = 300$, how much does the firm produce in the short run?

Question C.26
Production function $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$ with $w_1 = 4,\ w_2 = 1$.

a. Does the production function exhibit increasing, decreasing, or constant returns to scale?

b. Find the conditional factor demands.

c. Derive the cost function $c(y)$.

d. If the firm is a price-taker and the output price is $p = 600$, write the firm's profit function.

e. What output does the firm choose at $p = 600$?

Question C.27
A firm with $f(x_1, x_2) = x_1^{1/2} x_2^{1/2}$ faces $w_1 = 4,\ w_2 = 1$.

a. Does this production function have increasing, decreasing, or constant returns to scale?

b. Write an expression for the slope of the isoquants.

c. What is the slope of its isocosts?

d. Find the cost-minimizing (x_1, x_2) for output y.

e. Derive the cost function $c(y)$.

Question C.28
A firm with $f(x_1, x_2) = x_1^{1/2} x_2^{1/2}$ faces $w_1 = \frac{1}{2},\ w_2 = \frac{1}{2}$.

a. What is the marginal product of x_1? Does the firm exhibit diminishing marginal product for x_1?

b. Find the conditional factor demands for output y.

c. Derive the cost function $c(y)$.

Cournot (and Monopoly) Problems

Question C.29
Suppose market demand $q = 1000 - 10p$ and firms have cost function $c(q) = 10q$.

a. Derive the inverse demand function.

b. For a monopolist, write the profit function.

c. Find the monopolist's optimal quantity and price.

d. If there are two firms competeing in cournot competition, write firm 1's profit.

e. Derive firm 1's best response.

f. Leveraging symmetry, find the Nash equilibrium quantities.

g. What are market quantity and price in equilibrium?

Question C.30
Two firms have cost $c(q) = 5q$ and face inverse demand $p = 29 - Q$.

a. Write firm 1's profit function as a function of q_1, q_2.

b. Find firm 1's best response function.

c. Leverage symmetry to determine the Nash equilibrium.

Question C.31
A monopolist has cost $c(y) = 2y^2 + 400y + 10$ and demand is $q(p) = 1000 - p$.

a. Does this function have decreasing, increasing, or contant, marginal cost?

b. Write profit as a function of y.

c. Find the profit-maximizing y and the price.

d. Show that demand is elastic at that price.

Question C.32
Suppose demand is $Q = 3000 - p$ and firms have zero costs $c(q) = 0$.

a. Suppose there is one firm in the market (a monopolist) write the profit function.

b. What quantity maximizes profit for this monopolist?

c. With two firms producing q_1, q_2 and comepeting in cournot competition, write firm 1's profit.

d. Derive firm 1's best response function.

e. Find Nash equilibrium quantities (leverage symmetry).

Question C.33
Suppose demand is $q = 1000 - 10p$ and each firm has cost function $c(q) = 10q$.

a. What is the inverse demand?

b. Suppose there is one firm in the market (a monopolist) write the profit function.

c. Find the monopolist's optimal q and p?

d. Now suppose there are two firms comepeting in Cournot competition, write firm 1's profit.

e. Derive firm 1's best response.

f. Leverage symmetry to find the Nash equilibrium.

Question C.34
Suppose demand is $Q = 500 - p$ and firms have cost $c(q) = 200q$.

a. What is the inverse demand?

b. Suppose there is one firm in the market (a monopolist) write the profit function.

c. Find the monopolist's optimal q and p?

d. Now suppose there are two firms comepeting in Cournot competition, write firm 1's profit.

e. Derive firm 1's best response.

f. Leverage symmetry to find the Nash equilibrium.

A Few Additional Problems

Question C.35 A monopolist serves types of consumers with inverse demand functions:

$$p_A = 120 - 3q_A, \qquad p_B = 96 - 3q_B,$$

where q_A and q_B are the quantities sold to groups A and B, respectively. Cost is $C(q) = 12q$.

a. Find the profit-maximizing quantities q_A^*, q_B^* and corresponding prices p_A^*, p_B^*.

b. Compute the resulting maximum profit.

Question C.36 A monopolist sells two goods to two consumers with the following willingess to pay. The monopolist has zero cost.

	Good 1	Good 2
Consumer A	30	10
Consumer B	10	30

a. If the monopolist sets separate prices for good 1 and good 2 find the prices it should set to maximize profit and compute that profit.

b. If the monopolist sells only the bundle, find the price it should choose to maximize profit and compute that profit.

c. Which strategy yields the higher profit?

Question C.37 A firm is a price taker and faces a market price $p = 50$. Suppose their short run cost is $c_{sr}(q) = 20q + q^2 + 80$ and their long run cost is $c_{lr}(q) = 10q + q^2$.

a. In the short run find the profit-maximizing output and the profit.

b. In the long run find the profit-maximizing output and profit.

c. Explain briefly (one sentence) why long-run profit differs from short-run profit.

C.1 Solutions

Solution C.1

a. decreases, price

b. convex

Solution C.2

a. Decreases

b. income, increase

c. MRS (Marginal Rate of Substitution)

Solution C.3

a. More

b. Inferior

c. Monotonic

Solution C.4

a. increases

b. decreases

c. convex

Solution C.5

a. decreases, income

b. complete

Solution C.6

a. increases

b. monotonic

Solution C.7

a. yes

b. $a \sim a, b \sim b, c \sim c, a \sim b$

c. $a \succ c, b \succ c$

d. $a \sim b \succ c$

e. For example, $u(a) = 2, u(b) = 2, u(c) = 0$

Solution C.8

a. yes

b. $a \succ b, a \succ c, b \succ c$

c. $a \sim a, b \sim b, c \sim c$

d. $a \succ b \succ c$

e. For example, $u(a) = 3, u(b) = 2, u(c) = 1$

Solution C.9

a. $p_1x_1 + p_2x_2 = m$

b. $-\frac{p_1}{p_2}$

c. $-\frac{2}{3}$

d. $-\frac{x_2}{x_1} = -\frac{p_1}{p_2}$

e. $x_1 = \frac{\frac{1}{2}m}{p_1}, x_2 = \frac{\frac{1}{2}m}{p_2}$

f. $(5, 10)$

g. Normal

h. Neither

Solution C.10

a. $-\frac{5}{x_1}$

b. $-\frac{p_1}{p_2}$

c. $x_1 = 5, x_2 = 5$

d. $p_1 = 10$

Solution C.11

a. $100x_1 + 100x_2 = 1200$

b. $(6, 6)$

c. Decreases by 4 to $x_1 = 2$

d. $\tilde{m} = 2400$

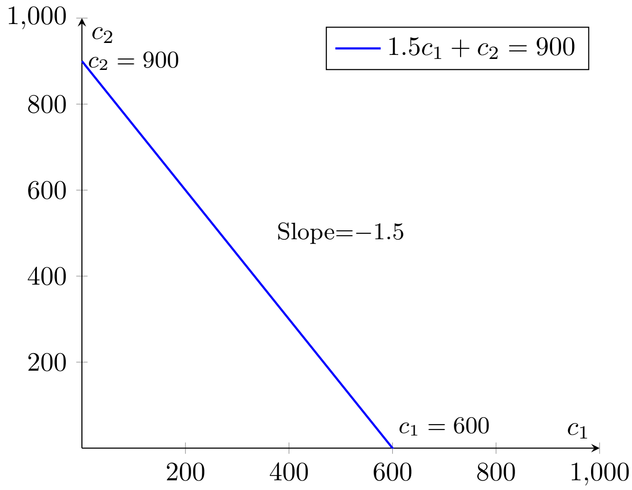
e. $(4, 12)$

f. Decrease of 2 due to substitution effect and decrease of 2 due to income effect.

Solution C.12

a. $600, 900$

b.



c. $300, 450$

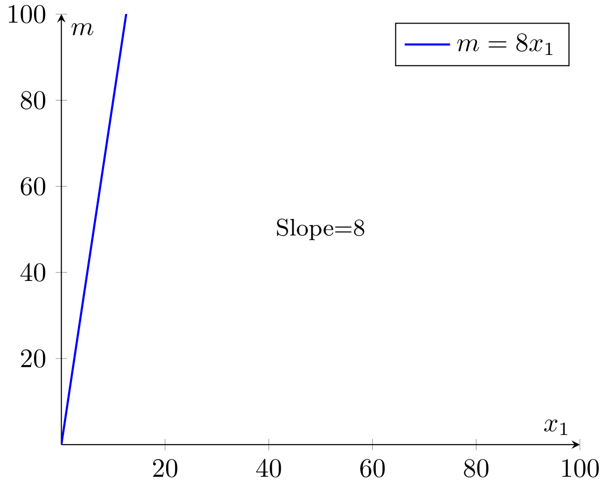
d. Borrower

e. Better-Off

Solution C.13

a. $p_1x_1 + p_2x_2 = m$

b. $\frac{m}{p_1+4p_2}$

c. 

Solution C.14

a. $(1+r)c_1 + c_2 = (1+r)600 + 1500$

b. $(1000, 1000)$

c. Borrower

d. Borrower

Solution C.15

a. $4x_1 + 2x_2 = 20$

b. -1

c. $(0, 10)$

d. $(5, 0)$

e. $\tilde{m} = 50$

f. All 10 is due to substitution.

Solution C.16

a. $x_1 + x_2 = 10$

b. $-\frac{x_2}{x_1}$

c. $x_1 = 5$

d. Seller

Solution C.17

a. $x_1 + 6x_2 = 900$

b. 100

c. Decrease of 40

d. 0 change due to substitution, decrease of 40 due to income

Solution C.18

a. $2x_1 + 4x_2 = 30$

b. $(15, 0)$

c. Buyer

d. Buyer

Solution C.19

a. $p_1x_1 + p_2x_2 = m$

b. $x_1 = 2x_2$

c. $x_1 = \frac{m}{p_1+\frac{1}{2}p_2}, x_2 = \frac{1}{2}\frac{m}{p_1+\frac{1}{2}p_2}$

Solution C.20

a. cost

b. decreasing

c. inelastic

Solution C.21

a. p$p = 4$, $Q = 40$.

b. $\varepsilon = -4$.

c. $p = 4.5,\ Q = 20$.

d. DWL $= 25$.

Solution C.22

a. $p = 100,\ Q = 100$.

b. $\epsilon = -2$.

c. Q decreases by 2%.

d. $p = 125$, $Q = 50$.

e. DWL $= 1875$.

Solution C.23

a. $p = 9,\ Q = 72$.

b. $\epsilon = -\frac{1}{8}$, inelastic.

c. $p = 17,\ Q = 64$.

d. Consumers pay \$8 more, producers receive \$1 less.

e. DWL $= 36$.

Solution C.24

a. $p = 100,\ Q = 200$.

b. CS $= 2500$.

c. $p = 110,\ Q = 120$.

d. DWL $= 2000$.

e. Revenue $G = t(200 - 1.6t)$ is maximized at $t = 62.5$.

Solution C.25

a. $x_1 = x_2 = y^{3/2}$.

b. $c(y) = 2\,y^{3/2}$.

c. $x_2 = 1$ requires $x_1 = y^3$, so $c(y) = y^3 + 1$.

d. Maximize $300y - (y^3 + 1)$. This is maximizes at $y = 10$.

Solution C.26

a. Decreasing returns to scale.

b. $x_1 = \frac{1}{2}y^{3/2},\ x_2 = 2y^{3/2}$.

c. $c(y) = 4y^{3/2}$.

d. $\pi(y) = 600y - 4y^{3/2}$.

e. $y = 10000$.

Solution C.27

a. Constant returns to scale.

b. $-\dfrac{x_2}{x_1}$.

c. -4.

d. $x_1 = \frac{y}{2},\ x_2 = 2y$.

e. $c(y) = 4y$.

Solution C.28

a. $MP_1 = \frac{1}{2}\frac{x_2^{1/2}}{x_1^{1/2}}$; yes, it is diminishing.

b. $x_1 = x_2 = y$.

c. $c(y) = \frac{1}{2}y + \frac{1}{2}y = y$.

Solution C.29

a. $p = \dfrac{1000 - Q}{10}$.

b. $\pi(q) = q\,\dfrac{1000 - q}{10} - 10q$.

c. $q = 450,\ p = 55$.

d. $\pi_1 = q_1\,\dfrac{1000 - q_1 - q_2}{10} - 10q_1$.

e. $q_1 = \dfrac{900 - q_2}{2}$.

f. $q_1 = q_2 = 300$.

g. $Q = 600,\ p = 40$.

Solution C.30

a. $\pi_1 = q_1(29 - (q_1 + q_2)) - 5q_1$.

b. $q_1 = 12 - \frac{1}{2}q_2$.

c. $q_1 = q_2 = 8$.

Solution C.31

a. Increasing marginal cost.

b. $\pi(y) = (1000 - y)y - (2y^2 + 400y + 10)$.

c. $y = 100,\ p = 900$.

d. $\epsilon = -\frac{p}{1000-p} = -\frac{900}{100} = -9$ (elastic).

Solution C.32

a. $\pi(q) = q\,(3000 - q)$.

b. $q = 1500$.

c. $\pi_1 = q_1\,(3000 - q_1 - q_2)$.

d. $q_1 = \frac{3000-q_2}{2}$.

e. $q_1 = q_2 = 1000$.

Solution C.33

a. $p = \dfrac{1000 - Q}{10}$.

b. $\pi(q) = q\,\dfrac{1000 - q}{10} - 10q$.

c. $q = 450,\ p = 55$.

d. $\pi_1 = q_1\,\dfrac{1000 - q_1 - q_2}{10} - 10q_1$.

e. $q_1 = 450 - \frac{1}{2}q_2$.

f. $q_1 = q_2 = 300$.

Solution C.34

a. $p = 500 - Q$.

b. $\pi(q) = q\,(500 - q) - 200q$.

c. $q = 150,\ p = 350$.

d. $\pi_1 = q_1\,(500 - q_1 - q_2) - 200q_1$.

e. $q_1 = 150 - \frac{1}{2}q_2$.

f. $q_1 = q_2 = 100$.

Solution C.35

a. $\pi(q_A, q_B) = (120 - 3q_A)q_A + (96 - 3q_B)q_B - 12(q_A + q_B)$.

b. $q_A^* = 18$, $q_B^* = 14$. $p_A^* = 66$, $p_B^* = 54$

c. Maximum profit: $\pi^* = 1560$.

Solution C.36

a. Optimal separate prices: $p_1 = p_2 = 30$. Profit $= 30 + 30 = 60$.

b. Optimal bundle price: $p = 40$.

c. Bundling yields the higher profit.

Solution C.37

a. $q = 15$ and profit is 145.

b. $q = 20$ and profit is 400

c. Long-run profit is higher because it can adjust it's use of any fixed inputs to the optimal level, leading to lower cost for the same output.

References