Cost functions:

$c(q_i) = 10q_i$

Inverse demand:

$p = 100 - Q$

Suppose there are two firms $1, 2$

$$\pi_1(q_1, q_2) = q_1(100 - (q_1 + q_2)) - 10q_1$$

$$\pi_2(q_1, q_2) = q_2(100 - (q_1 + q_2)) - 10q_2$$

0.1 Best Responses

The **best response function** tells us what each firms optimal quantity is, given the other firm's quantity.

To find the best response of firm 1, we look for the optimal q_1 as a function of q_2. Maximize the profit function in terms of q_1.

Let's first simplify the profit function:

$$\pi_1(q_1, q_2) = q_1(100 - (q_1 + q_2)) - 10q_1$$

$$= 100q_1 - q_1^2 - q_1q_2 - 10q_1$$

$$= 90q_1 - q_1^2 - q_1q_2$$

Where is this maximized in terms of q_1? We look for where the derivative with respect to q_1 is zero.

$$\frac{\partial\left(90q_1 - q_1^2 - q_1q_2\right)}{\partial q_1} = 0$$

$$90 - 2q_1 - q_2 = 0$$

Solve this for q_1 to get the **best response function:**

$$q_1 = 45 - \frac{1}{2}q_2$$

For instance, suppose firm 1 things $q_2 = 10$.

Best response is:

$$q_1 = 45 - \frac{1}{2}(10) = 40$$

Their profit if $q_2 = 10$:

$$q_1 (100 - (q_1 + 10)) - 10q_1$$

$$\begin{pmatrix} 10 & 700 \\ 20 & 1200 \\ 30 & 1500 \\ 40 & 1600 \\ 50 & 1500 \\ 60 & 1200 \end{pmatrix}$$

This confirms that $q_1 = 40$ is the optimal choice of quantity if $q_2 = 10$.

Firm 2's best response:

$$\pi_2 (q_1, q_2) = q_2 (100 - (q_1 + q_2)) - 10q_2$$

$$= 100q_2 - q_1 q_2 - q_2^2 - 10q_2$$

$$= 90q_2 - q_1 q_2 - q_2^2$$

Firm 2 maximizes this profit given the choice q_1.

$$\frac{\partial \left(90q_2 - q_1 q_2 - q_2^2\right)}{\partial q_2} = 0$$

$$90 - q_1 - 2q_2 = 0$$

$$2q_2 = 90 - q_1$$

$$q_2 = 45 - \frac{1}{2} q_1$$

0.2 Best response functions

$$q_1 = 45 - \frac{1}{2}q_2$$

$$q_2 = 45 - \frac{1}{2}q_1$$

A **nash** equilibrium is a set of quantities that simultaneously best responses to eachother.

Let's try a set quantities $q_1 = 40, q_2 = 10$. Is $(40, 10)$ a Nash equilibrium?

$q_1 = 40$ is a best response to $q_2 = 10$. $40 = 45 - \frac{1}{2}(10)$.

$q_2 = 10$ is **not a best response to** $q_1 = 40$**.** The actual best response is given by firm 2's best response function:

$$q_2 = 45 - \frac{1}{2}(40) = 25$$

0.3 Finding the actual equilibrium:

The actual nash equilibrium, requires finding q_1, q_2 that simultaneously solve:

$$Solve[\{q_1 == 45 - \frac{1}{2}q_2, q_2 == 45 - \frac{1}{2}q_1\}, \{q_1, q_2\}]$$

$$\{\{q_1 \to 30, q_2 \to 30\}\}$$

It will alwas be true in cournot games like this, that if the firms have the same cost function, the nash equilibrium will involve them **choosing the same quantity**. The nash equilibrium will be **symmetric**.

We can leverage this to simplify finding the Nash equilibrium.

0.4 Leveraging Symmetry

Take any firm's best response function:

$$q_1 = 45 - \frac{1}{2}q_2$$

$$q_2 = 45 - \frac{1}{2}q_1$$

If we impose that $q_1 = q_2 = q$

$$q = 45 - \frac{1}{2}q$$

What is a quantity, that if we both choose it, it is a best response to iteself.

Let's solve this:

$$\frac{1}{2}q + q = 45$$

$$\frac{3}{2}q = 45$$

$$q = \frac{2}{3}45 = 15 * 2 = 30$$

The nash equilibrium of this game is $q^* = 30$.

0.5 One More Example

Inverse demand where Q is the market demand $Q = q_1 + q_2$

$p(Q) = 25 - Q$

Cost function of each firm:

$c(q_i) = 5q_i$

Suppose there are two firms.

a) Write down firm 1's profit function as a function of q_1 and q_2.

$$\pi_1(q_1, q_2) = q_1(25 - (q_1 + q_2)) - 5q_1$$

Simplify this:

$$\pi_1(q_1, q_2) = q_1(25 - q_1 - q_2) - 5q_1$$

$$= 25q_1 - q_1^2 - q_1q_2 - 5q_1$$

$$= 20q_1 - q_1^2 - q_1q_2$$

b) What is firm 1 profit if they choose $q_1 = 10$ and firm $q_2 = 20$

$$20\,(10) - 10^2 - 10 * 20 \quad = \quad -100$$

c) What is firm 1's best response function this should give q_1 as a function of q_2.

$$\pi_1\,(q_1, q_2) = 20q_1 - q_1^2 - q_1 q_2$$

Find:

$$\frac{\partial\left(20q_1 - q_1^2 - q_1 q_2\right)}{\partial q_1} = 0$$

$$20 - 2q_1 - q_2 = 0$$

$$20 - q_2 = 2q_1$$

$$q_1 = \frac{20 - q_2}{2} = 10 - \frac{1}{2}q_2$$

$$q_1 = 10 - \frac{1}{2}q_2$$

This is firm 1's best response function.

c) What q_1 maximizes firm 1's profit when $q_2 = 20$

$$q_1 = 10 - \frac{1}{2}\,(20)$$

$$q_1 = 0$$

d) What is the nash equilibrium of this game. (impose symmetry)

$$q_1 = 10 - \frac{1}{2}q_2$$

Assume $q_1 = q_2 = q$

$$q = 10 - \frac{1}{2}q$$

Solve for q:

$$\frac{3}{2}q = 10$$

$$q = \frac{20}{3} = 6.666...$$

0.6 Solving the game with many firms.

N firms.

$p(Q) = 25 - Q$

Cost function of each firm:

$c(q_i) = 5q_i$

Let's set up the profit function of firm i.

$$\pi_i = q_i (25 - Q) - 5q_i$$

Let's use Q_{-i} to refer to the sum of the quantities that "aren't mine" (in Nashville "y'alls quantity".

$$Q_{-i} = Q - q_i$$

This lets use write (instead of this)

$$\pi_i = q_i (25 - q_1 + q_2 + q_3 ... + q_N) - 5q_i$$

We can write this:

$$\pi_i (q_i, Q_{-i}) = q_i (25 - (q_i + Q_{-i})) - 5q_i$$

$$\pi_i (q_i, Q_{-i}) = 25q_i - q_i^2 - q_i Q_{-i} - 5q_i$$

$$= 20q_i - q_i^2 - q_i Q_{-i}$$

What is the best response of firm i to Q_{-i}?

$$\frac{\partial\left(20q_i - q_i^2 - q_i Q_{-i}\right)}{\partial q_i} = 20 - 2q_i - Q_{-i}$$

This is maximized where the derivative is zero:

$$20 - 2q_i - Q_{-i} = 0$$

Solve for q_i

$$20 - Q_{-i} = 2q_i$$

$$q_i = 10 - \frac{1}{2}Q_{-i}$$

Technically a Nash equilibrium is a solution to N equations $\left\{q_i = 10 + \frac{1}{2}Q_{-i}\right\}_{i=1}^{N}$ however we know that the solution will be symmetric. Se we impose symmetry on any of the best response functions and it will be the Nash equilibrium.

$q_1 = q_2 = q_3 = ... = q_N = q$

If we all choose the same quanitity q then $Q = Nq$. $Q_{-i} = Nq - q = (N-1)\,q$

If there are $N-1$ firms who aren't me and they all choose the same quantity then $Q_{-i} = (N-1)\,q$

$$q = 10 - \frac{1}{2}(N-1)\,q$$

To find the equilibrium, solve this for q:

$$q + \frac{1}{2}(N-1)\,q = 10$$

$$\left(1 + \frac{1}{2}(N-1)\right)q = 10$$

In equilibrium:

$$q = \frac{10}{1 + \frac{1}{2}(N-1)}$$

Market quantity in equilibrium:

$$Q = N \frac{10}{1 + \frac{1}{2}(N-1)}$$

Market price:

$$p = 25 - N \frac{10}{1 + \frac{1}{2}(N-1)}$$

N	p	Q
1	10.	15.
2	13.3333	11.6667
5	16.6667	8.33333
10	18.1818	6.81818
100	19.802	5.19802
1000	19.98	5.01998