Two firms compete in “cournot competition”.

$N = 2$

Cost functions:

$c(q_i) = 10q_i$

Inverse demand:

$p = 100 - Q$

Set up the profit functions of the firms:

$$\pi_1(q_1, q_2) = q_1(100 - (q_1 + q_2)) - 10q_1$$

$$= q_1(100 - q_1 - q_2) - 10q_1$$

$$= 100q_1 - q_1^2 - q_1q_2 - 10q_1$$

$$= 90q_1 - q_1^2 - q_1q_2$$

$$\pi_2(q_2, q_1) = q_2(100 - (q_1 + q_2)) - 10q_2$$

$$= 90q_2 - q_2^2 - q_1q_2$$

0.1 Best Responses

What is firm 1’s choice?

Instead of optimal quantity, like we have in monopoly, here we have a **best response function** an optimal q_1 for any choice of q_2.

To find the best response function find where the profit function of firm one has zero slope with respect to q_1.

$$\frac{\partial(q_1(100 - (q_1 + q_2)) - 10q_1)}{\partial q_1} = 0$$

Above, we simplified the profit function to this:

$$\frac{\partial\left(90q_1 - q_1^2 - q_1q_2\right)}{\partial q_1} = 0$$

$$90 - 2q_1 - q_2 = 0$$

Solve for q_1 to get the optimal q_1.

$$90 - q_2 = 2q_1$$

$$q_1 = 45 - \frac{1}{2}q_2$$

This is the **best response function** for firm 1.

For example, if $q_2 = 20$, firm one's optimal choice of q_1 is $45 - \frac{1}{2}20 = 35$.

$$q_1 = 35$$

Let's confirm this:

$$\pi(q_1, 20) = q_1(100 - (q_1 + 20)) - 10q_1$$

$$\begin{pmatrix} q_1 & \pi_1 \\ 5 & 325 \\ 10 & 600 \\ 15 & 825 \\ 20 & 1000 \\ 25 & 1125 \\ 30 & 1200 \\ 35 & 1225 \\ 40 & 1200 \\ 45 & 1125 \\ 50 & 1000 \end{pmatrix}$$

Best response for firm 2 is identical (try this if you want by maximizing π_2 with respect to q_2)

$$q_2 = 45 - \frac{1}{2}q_1$$

Best response functions:

$$q_1 = 45 - \frac{1}{2}q_2, q_2 = 45 - \frac{1}{2}q_1$$

Nash Equilibium

Is this a reasonable prediction for the game?

$$(q_1, q_2) = (35, 20)$$

We know that $q_1 = 35$ is a best response to $q_2 = 20$.

However, what is the best response to 35?

$$q_2 = 45 - \frac{1}{2}(35.0) = 27.5$$

This is not a stable prediction for what would happen in this game since firm 2 has an incentive to change their behavior.

What is the **nash equilibrium.** What is the set of pairs (q_1, q_2) such that q_1 is a best response to q_2 and q_2 is a best response to q_1?

$$\{\{q_1 \to 30, q_2 \to 30\}\}$$

This is a symmetric nash equilibrium (the firms choose the same quantity). All of the games we look at will have a symmetric nash equilibrium because we will assume the have the same cost function.

0.2 Leveraging Symmetry

Since we know the nash equilibrium has to be symmetric, we can simplify solving it by looking at just one of the firm's best response functions:

$$q_1 = 45 - \frac{1}{2}q_2$$

Assume $q_1 = q_2 = q$

$$q = 45 - \frac{1}{2}q$$

If we solve this for q:

$$\frac{3}{2}q = 45$$

$$q = \frac{45}{3}2 = 15 * 2 = 30$$

1. Set up any firm's profit function.
2. Find their best response function.
3. Leverage symmetry $q_1 = q_2 = q$
4. Solve for q this is the nash equilibrium.

0.3 One More Example

Inverse demand where Q is the market demand $Q = q_1 + q_2$

$p(Q) = 25 - Q$

Cost function of each firm:

$c(q_i) = 5q_i$

a) Firm firm 1's profit function.

$$\pi_1 = q_1 (25 - (q_1 + q_2)) - 5q_1$$

$$= 25q_1 - q_1^2 - q_1 q_2 - 5q_1$$

$$= 20q_1 - q_1^2 - q_1 q_2$$

b) Find firm 1's best response function (q_1 as a function of q_2)

$$\frac{\partial \left(20q_1 - q_1^2 - q_1 q_2\right)}{\partial q_1} = 0$$

$$20 - 2q_1 - q_2 = 0$$

$$20 - q_2 = 2q_1$$

$$q_1 = 10 - \frac{1}{2} q_2$$

c) What is the optimal quantity for firm 1 if firm two chooses $q_2 = 10$

$$q_1 = 10 - 5 = 5$$

d) Leverage symmetry to find the nash equilibrium.

We know $q_1 = q_2 = q$ in nash equilibrium:

$$q = 10 - \frac{1}{2}q$$

$$\frac{3}{2}q = 10$$

$$q = \frac{20}{3} \approx 6.6667$$

e) What price do the firms get in equilibrium?

The price they get is determined by the inverse demand:

$$p = \left(25 - \left(\frac{20}{3} + \frac{20}{3}\right)\right) = \frac{35}{3}$$

f) What profit do the firms earn in equilibrium?

$$\pi\left(\frac{20}{3}, \frac{20}{3}\right) = 20\left(\frac{20}{3}\right) - \left(\frac{20}{3}\right)^2 - \left(\frac{20}{3}\right)\left(\frac{20}{3}\right)$$

$$20\left(\frac{20}{3}\right) - \frac{20^2}{3^2} - \frac{20}{3}\frac{20}{3} = \frac{400}{9}$$

0.4 Solving the game with many firms.

Suppose we have N firms:

$p\left(Q\right) = 25 - Q$

Cost function of each firm:

$c\left(q_i\right) = 5q_i$

$$\pi_i = q_i\left(25 - \left(q_i + Q_{-i}\right)\right) - 5q_i$$

What is the best response of $firm\ i$ to the a Q_{-i}?

$$q_i = 10 - \frac{1}{2}Q_{-i}$$

Leverage symmetry:

$q_1 = q_2 = q_3 = ... = q_N = q$

$$q_i = 10 - \frac{1}{2}Q_{-i}$$

$$q = 10 - \frac{1}{2}\left((N-1)\,q\right)$$

$$q = \frac{20}{N+1}$$

What is the market quantity:

$$Q = Nq = \frac{N}{N+1}20$$

Market price:

$$p = 25 - \frac{N}{N+1}20$$

N	q	Q	p
1	10.	10.	15.
2	6.66667	13.3333	11.6667
5	3.33333	16.6667	8.33333
10	1.81818	18.1818	6.81818
100	0.19802	19.802	5.19802
1000	0.01998	19.98	5.01998