1 Production

$f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$

$f(4,4) = \sqrt{4} + \sqrt{4} = 4$

$f(5,4) = \sqrt{5.0} + \sqrt{4.0} = 4.23607$

$4.23607 - 4 = 0.23606$

$f(6,4) = \sqrt{6.0} + \sqrt{4.0} = 4.44949$

$4.44949 - 4.23607 = 0.21342$

Notice that the *extra* output you get by increasing x_1 decreases.

This is because of **decreasing marginal product.**

$$MP_1 = \frac{\partial\left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)}{\partial x_1} \quad = \quad \frac{1}{2\sqrt{x_1}}$$

Decreasing marginal product for x_1 since MP_1 is decreasing in x_1 (since x_1 only appears in the denominator).

1.1 Another Example:

$x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$

$$MP_1 = \frac{\partial\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)}{\partial x_1}$$

$$\frac{\partial\left(10x_1^{\frac{1}{2}}\right)}{\partial x_1} = 10\frac{1}{2}x_1^{-\frac{1}{2}}$$

$$\frac{\partial\left(100x_1^{\frac{1}{2}}\right)}{\partial x_1} = 100\frac{1}{2}x_1^{-\frac{1}{2}}$$

$$MP_1 = \frac{\partial\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)}{\partial x_1} = x_2^{\frac{1}{2}}\frac{1}{2}x_1^{-\frac{1}{2}} = \frac{1}{2}\frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}}$$

$$MP_2 = \frac{1}{2}\frac{x_1^{\frac{1}{2}}}{x_2^{\frac{1}{2}}}$$

Diminishing Marginal Product for input 1 and 2.

1.2 Returns to Scale

$$f(1,1) = \sqrt{1}\sqrt{1} = 1$$

$$f(2,2) = \sqrt{2}\sqrt{2} = 2$$

$$f(3,3) = \sqrt{3}\sqrt{3} = 3$$

$$f(4,4) = \sqrt{4}\sqrt{4} = 4$$

As we scale production by increasing both inputs, extra output does not diminish. (we get one extra output each time).

While we have diminishing marginal, we don't have decreasing returns to scale.

Marginal Product: amount of extra production you get when you increase one input.

Returns to Scale: amount of extra production you get when you increase all inputs.

Increasing, Decreasing, Constant returns to scale.

Formally, we check this by asking how output changes when we scale input by $t > 1$.

Constant (Linear): $f(tx_1, tx_1) = tf(x_1, x_2)$

Double both input and you get exactly double the output.

Decreasing: $f(tx_1, tx_2) < tf(x_1, x_2)$

Double both inputs and you get less than double the output.

Increasing: $f(tx_1, tx_2) > tf(x_1, x_2)$

Double both inputs and you get more than double the output.

$$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

This seems to have linear returns to scale:

$$f(1,1) = \sqrt{1}\sqrt{1} = 1$$

$$f(2,2) = \sqrt{2}\sqrt{2} = 2$$

$$f(4,4) = \sqrt{4}\sqrt{4} = 4$$

Each time we double the inputs, we get exactly double the output.

$$f(tx_1, tx_2) = (tx_1)^{\frac{1}{2}} (tx_2)^{\frac{1}{2}} = t^{\frac{1}{2}} x_1^{\frac{1}{2}} t^{\frac{1}{2}} x_2^{\frac{1}{2}} = t x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} = t f(x_1, x_2)$$

1.3 In summary:

$$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

Diminishing marignal product for inputs, but constant (linear) returns to scale.

$$f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$$

$$f(tx_1, tx_2) = (tx_1)^{\frac{1}{2}} + (tx_2)^{\frac{1}{2}} = t^{\frac{1}{2}} x_1^{\frac{1}{2}} + t^{\frac{1}{2}} x_2^{\frac{1}{2}} = t^{\frac{1}{2}} \left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}} \right) = t^{\frac{1}{2}} f(x_1, x_2)$$

Since $t^{\frac{1}{2}} < t$ we have $f(tx_1, tx_2) < tf(x_1, x_2)$. Decreasing returns to scale.

Example For each of these, do they have increasing, decrease or constant returns to scale?

a) $f(x_1, x_2) = x_1 + 2x_2$

$$f(tx_1, tx_2) = tx_1 + 2tx_2 = t(x_1 + 2x_2) = tf(x_1, x_2)$$

Constant/Linear Returns to scale.

$$f(1,1) = 1 + 2 * 1 = 3$$

$$f(2,2) = 2 + 2 * 2 = 6$$

b) $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{2}}$

$$f(tx_1, tx_2) = (tx_1)^{\frac{1}{4}} (tx_2)^{\frac{1}{2}} = t^{\frac{1}{4}} x_1^{\frac{1}{4}} t^{\frac{1}{2}} x_2^{\frac{1}{2}} = t^{\frac{3}{4}} \left(x_1^{\frac{1}{4}} x_2^{\frac{1}{2}} \right) = t^{\frac{3}{4}} f(x_1, x_2)$$

Notice that $\frac{3}{4}$ is the sum of the exponents. As a rule of thumb, a cobb douglass production function has decreasing RTS if the sum of the exponents is less than one. Constant if the sum is 1. Increasing if the sum is more than 1.

$$f(1,1) = 1^{\frac{1}{4}} 1^{\frac{1}{2}} = 1$$

$$f(2,2) = 2.0^{\frac{1}{4}} 2.0^{\frac{1}{2}} = 1.68179$$

When inputs doubled, output less-than-doubled. Decreasing returns to scale.

c) $f(x_1, x_2) = min\{x_1, x_2\}$

$$min\{2,3\} = 2$$

$$min\{4,6\} = 4$$

$$f(tx_1, tx_2) = min\{tx_1, tx_2\} = tmin\{x_1, x_2\} = tf(x_1, x_2)$$

Constant/Linear returns to scale.

$$f(1,1) = min\{1,1\} = 1$$

$$f(2,2) = min\{2,2\} = 2$$

Constant/Linear Returns to scale.

2 Cost Minimization

We assume the goal of a firm is to maximize profit.

2.1 Motivation- Profit Max

Fromally write down the profit function of a firm.

Firms have a technology $f(x_1, x_2)$.

They pick a set of inputs, that produces output, they sell the output for profit.

Profit is revenue minus **cost**.

Cost of inputs are w_1, w_2.

Cost of an input bundle (x_1, x_2) is:

$$w_1 x_1 + w_2 x_2$$

Revenue from using (x_1, x_2) depends on how much you produce $f(x_1, x_2)$ and the **price that you get for selling those units**. In most scenariors, the price you get depends on how much you try to sell. $p(f(x_1, x_2))$. The revenue of a firm:

$$f(x_1, x_2)\, p(f(x_1, x_2))$$

Profit function π :

$$\pi(x_1, x_2) = f(x_1, x_2)\, p(f(x_1, x_2)) - (w_1 x_1 + w_2 x_2)$$

If we assume price is fixed:

$$\pi(x_1, x_2) = pf(x_1, x_2) - (w_1 x_2 + w_2 x_2)$$

Let's plug in a production function $x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$ and $w_1 = 1$ $w_2 = 1$,$p = 10$

$$\pi(x_1, x_2) = 10 x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (x_1 + x_2)$$

$$\frac{\partial \left(10 x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (x_1 + x_2)\right)}{\partial x_1} = 0$$

$$\frac{\partial \left(10 x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (x_1 + x_2)\right)}{\partial x_2} = 0$$

$$\frac{10\sqrt[3]{x_2}}{3x_1^{2/3}} - 1 = 0$$

$$\frac{10\sqrt[3]{x_1}}{3x_2^{2/3}} - 1 = 0$$

$$\left\{\left\{x_1 \to \frac{1000}{27}, x_2 \to \frac{1000}{27}\right\}\right\}$$

Howmuch will the firm produce?

$$y = 11.1111$$

2.2 Profit Max Requires Cost Min

No matter a firm will produce *some* ***level*** of output y.

If you choose a input bundle that is not the cheapest way of producing y, then you can increase profit by choosing the cost minimizing input bundle for producing y.

Revenue won't change, but costs will decrease.

Maximizing profit requires minimizing costs.

2.3 Two Step Process for Profit Max

-We first find the cheapest way to produce *any* level of output y.

$-$Cost function $c(y)$ that tells us the cost of producing y in the cheapest possible way.

- Set up a profit function that only depends on y:

$$\pi(y) = yp(y) - c(y)$$