1 Production

$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$

Find the marginal product of each input. Does this production function have decreasing (diminishing) marginal product for each input?

$$MP_1 = \frac{\partial \left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \right)}{\partial x_1} =$$

$$MP_1 = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} = \frac{1}{2} \frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} = \frac{1}{2} \frac{\sqrt{x_2}}{\sqrt{x_1}}$$

Since the marginal product is decreasing in x_1 (since it only appears in the denominator), this has diminishing marginal product.

$$MP_2 = \frac{\partial \left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \right)}{\partial x_2} = \frac{1}{2} \frac{\sqrt{x_1}}{\sqrt{x_2}}$$

Since the marginal product (of x_2) is decreasing in x_2 (since it only appears in the denominator), this has diminishing marginal product.

1.1 Returns to Scale

$$f(1,1) = \sqrt{1}\sqrt{1} = 1$$

$$f(2,2) = \sqrt{2}\sqrt{2} = 2$$

$$f(4,4) = \sqrt{2}\sqrt{2} = 4$$

While this production function has diminishing marginal products, it appers that if you scale **both** inputs, the production does not become less and less efficient.

This production function has **linear (constant) returns to scale.**

linear (constant) returns to scale if for any $t > 1$:

$$f(tx_1, tx_2) = tf(x_1, x_2)$$

For example:

$$f(1,1) = 2, f(2,2) = 4$$

Increasing returns to scale if for any $t > 1$:

$$f(tx_1, tx_2) > tf(x_1, x_2)$$

For example:

$$f(1,1) = 2, f(2,2) = 5$$

Decreasing returns to scale if for any $t > 1$:

$$f(tx_1, tx_2) < tf(x_1, x_2)$$

For example:

$$f(1,1) = 2, f(2,2) = 3$$

1.2 Using the definition to prove returns to scale.

Does $x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ have constant returns to scale *everywhere*?

Let's try for one more example:

$$f(3,1) = \sqrt{3}\sqrt{1} = \sqrt{3}$$

$$f(6,2) = \sqrt{6}\sqrt{2} = \sqrt{3*2}\sqrt{2} = \sqrt{3}\sqrt{2}\sqrt{2} = 2\sqrt{3}$$

Is this true everywhere?

$$(tx_1)^{\frac{1}{2}} (tx_2)^{\frac{1}{2}} = f(tx_1, tx_2) = tf(x_1, x_2) = t\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)$$

$$(tx_1)^{\frac{1}{2}} (tx_2)^{\frac{1}{2}} = t^{\frac{1}{2}} x_1^{\frac{1}{2}} t^{\frac{1}{2}} x_2^{\frac{1}{2}} = tx_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

1.2.1 Example

For each of these, does the production function have dimishing marginal product? And does it have constant, increasing, or decreasing returns to scale?

a) $f(x_1, x_2) = x_1 + 2x_2$

$$MP_1 = \frac{\partial (x_1 + 2x_2)}{\partial x_1} = 1$$

$$MP_2 = \frac{\partial (x_1 + 2x_2)}{\partial x_2} = 2$$

Neither of these have dimishing marginal product.

$$f(1,1) = 1 + 2 = 3, f(2,2) = 2 + 4 = 6$$

$$f(3,1) = 3 + 2 * 1 = 5, f(6,2) = 6 + 2 * 2 = 10$$

b) $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{2}}$

$$MP_1 = \frac{\partial \left(x_1^{\frac{1}{4}} x_2^{\frac{1}{2}} \right)}{\partial x_1} = \frac{1}{4} x_1^{\frac{1}{4}-1} x_2^{\frac{1}{2}} = \frac{1}{4} x_1^{-\frac{3}{4}} x_2^{\frac{1}{2}} = \frac{1}{4} \frac{x_2^{\frac{1}{2}}}{x_1^{\frac{3}{4}}}$$

$$MP_2 = \frac{\partial \left(x_1^{\frac{1}{4}} x_2^{\frac{1}{2}} \right)}{\partial x_2} = \frac{1}{2} x_2^{-\frac{1}{2}} x_1^{\frac{1}{4}} = \frac{1}{2} \frac{x_1^{\frac{1}{4}}}{x_2^{\frac{1}{2}}}$$

$$f(1,1) = 1^{\frac{1}{2}} 1^{\frac{1}{4}} = 1, f(2,2) = 2^{\frac{1}{4}} 2^{\frac{1}{2}} = 2^{\frac{3}{4}} = 1.68179$$

Rule of thumb for **cobb doglass** production.

If exponent on a good is less than 1, you have diminishing marginal product.

If the sum of the exponents is less than 1, decreasing returns to scale.

If the sum of the exponents is greater than 1, increasing returns to scale.

If the sum of the exponents is 1, constant returns to scale.

Decreasing returns to scale.

c) $f(x_1, x_2) = x_1^{\frac{2}{3}} x_2^{\frac{2}{3}}$

Diminishing marginal product for both inputs, but increasing returns to scale.

d) $f(x_1, x_2) = min\{x_1, x_2\}$

$$min\{tx_1, tx_2\} = tmin\{x_1, x_2\}$$

$$min\{1, 2\} = 1, min\{2, 4\} = 2$$

Linear/Constant Returns to scale.

2 Cost Minimization

We assume the goal of a firm is to **maximize profit**.

Profit is made up of revenue minus their costs.

A firm who produces output y using production $f(x_1, x_2)$.

Revenue is y times the **price they can get** per unit of y they produce. py

Costs are x_1 times the cost of x_1 per unit w_1 plus the x_2 times the cost of x_2 per units w_2.

$$py - (w_1 x_1 + w_2 x_2)$$

The amount they produce y is deteremines by the production function:

$$\pi(x_1, x_2) = pf(x_1, x_2) - (w_1 x_1 + w_2 x_2)$$

Technically, the price p that you can get for your goods will depend on how much you want to sell $f(x_1, x_2)$. We should really make p a function of the output amount $f(x_1, x_2)$:

$$\pi(x_1, x_2) = p(f(x_1, x_2)) f(x_1, x_2) - (w_1 x_1 + w_2 x_2)$$

In markets each firm is so small, it is reasonable to think they take prices as fixed p. We call this the **price taking assumption**.

$$\pi(x_1, x_2) = pf(x_1, x_2) - (w_1 x_1 + w_2 x_2)$$

It is possible to maximize this function in one-go.

Let's make some assumptions:

$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$, $p = 10, w_1 = 1, w_2 = 1$

$$\pi(x_1, x_2) = 10x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - x_1 - x_2$$

$$10x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - x_1 - x_2$$

$$\frac{\partial\left(10x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - x_1 - x_2\right)}{\partial x_1} = 0$$

$$\frac{\partial\left(10x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - x_1 - x_2\right)}{\partial x_2} = 0$$

$$\left\{\left\{x_1 \to \frac{1000}{27}, x_2 \to \frac{1000}{27}\right\}\right\}$$

2.1 Profit Max Requires Cost Min

No matter what, at the maximum, the firm produces **some** output y^*.

at the maximum, the firm has to produce whaterver output it is producing in the cheapest possible way.

Profit maximization implies that firm is **minimizing cost**.

What is the cheapest way to produce any level of y?

$c(y)$

$$\pi(y) = py - c(y)$$