1 Cardinal Social Choice

$a \succ_1 b \succ_1 c$

$b \succ_2 a \succ_2 c$

$c \succ_3 a \succ_3 b$

Arrows says there is no way to make a choice that is always Pareto efficient, respects IIA.

What if we have more information? Like the cardinal utilities?

$a \succ_1 b \succ_1 c \quad U_1(a) = 10, U_1(b) = 5, U_1(c) = 4$

$b \succ_2 a \succ_2 c \quad U_2(a) = 5, U(b) = 10, U_2(c) = 4$

$c \succ_3 a \succ_3 b \quad U_3(a) = 2, U_3(b) = 1, U_3(c) = 1000$

Notice how useful this information is here. The fact that person 3 really likes c makes c a much more attractive outcome for society. In fact, relative to how much 3 likes outcome c, everything else looks almost like indifference.

As we will see, if we have this extra information, we can use it to get around Arrow's impossibility theorem.

1.1 Welfare Functions.

A welfare function is like a utility function but for the administrator. It takes in the utilities for everyone of an outcome and returns a number that represents "how good" that outcome is for society.

Remember the model where Alice and Bob have to clean the kitchen.

$O = \{ab, a, b, n\}$

$U_a(b) = 25, U_a(ab) = 12.5, U_a(a) = 10, U_a(n) = 5$

$U_b(b) = 10, U_b(ab) = 12.5, U_b(a) = 25, U_b(n) = 5$

1.2 Utilitarian Welfare

The utilitarian welfare measures how good an outcome is by averaging the individual utilities.

$W(b) = \frac{U_a(a)+U_b(a)}{2} = \frac{25+10}{2} = \frac{35}{2} = 17.5$

$W(a) = 17.5$

$W(ab) = 12.5$

$W(n) = 5$

To turn these into a social preference, if an outcome gets a higher score rank it higher. The social preferences induced by this welfare function are:

$$a \sim^* b \succ^* ab \succ^* n$$

To turn these into a social choice, make the social choice(s) the outcomes that get the highest welfare. The social choices are:a, b

1.3 Rawlsian Welfare Function

The Rawlsian welfare function favors fairness above all else. It calcuates the welfare as the minimum utility of any individual for a particular outcome.

$W(x) = min\{U_1(x), U_2(x), U_3(x), ..., U_n(x)\}$

$W(b) = min\{U_a(b), U_b(b)\} = min\{25, 10\} = 10$

$W(a) = min\{U_a(a), U_b(a)\} = min\{10, 25\} = 10$

$W(ab) = min\{U_a(ab), U_b(ab)\} = min\{12.5, 12.5\} = 12.5$

$W(n) = min\{U_a(n), U_b(n)\} = min\{5, 5\} = 5$

$$ab \succ^* a \sim^* b \succ^* n$$

The social choice is ab.

1.4 Compare Previous Two

Utilitarian welfare $W(10, 10) = W(20, 0) = W(0, 20)$

Rawlsian welfare $W(5, 5) > W(20, 0)$

Utilitarian only cares about "efficiency" and Rawlsian only cares about fairness.

1.5 Nash Welfare Function

Nash welfare function tends to balance efficiency and fairness.

$W(x) = U_1(x)^{\frac{1}{2}} * U_2(x)^{\frac{1}{2}}$

$W(x) = U_1(x)^{\frac{1}{n}} U_2(x)^{\frac{1}{n}} ...U_n(x)^{\frac{1}{n}}$

$W(b) = U_a(b)^{\frac{1}{2}} U_b(b)^{\frac{1}{2}} = 25^{\frac{1}{2}} 10^{\frac{1}{2}} = \sqrt{25}\sqrt{10} = 15.8114$

$W(a) = 10^{\frac{1}{2}} 25^{\frac{1}{2}} = \sqrt{250} = 15.8114$

$W(ab) = 12.5^{\frac{1}{2}} 12.5^{\frac{1}{2}} = \sqrt{12.5^2} = 12.5$

$W(n) = 5^{\frac{1}{2}} 5^{\frac{1}{2}} = 5$

$$a \sim^* b \succ^* ab \succ^* n$$

Social choice a, b

1.6 Welfare Function is a Non-Empty, Complete/Transitive, IIA

Does a welfare function always create a complete and transitive preference relation?

Yes.

Similarly, there will always be some outcome that gets the highest welfare from a set. That outcome will be the social choice. Any welfare function also is a **nonempty** social choice function.

Are they IIA?

A preference aggregation rule is IIA if we don't change the utilities of a and b, then the social preference over a and b won't change.

Suppose before we make any changes that $W(a) \geq W(b)$. If we don't change anything about idividual utilities of a and b then it will still be the case $W(a) \geq W(b)$ since $W(a)$ and $W(b)$ won't change.

Let's look again at utilitarian welfare. Let's hold the utilities for outcomes a and b fixed. And change other utilities. Will it still be that $a \sim^* b$? Yes, because if we don't change anyone's utilities for a and b then the $W(a)$ and $W(b)$ will be the same as before!

$W(b) = U_a(b) + U_b(b) = 25 + 10 = 35$

$W(a) = U_a(a) + U_b(a) = 10 + 25 = 35$

We still have $a \sim^* b$

So in summary, because a welfare calculation only looks at the utilities within a particular outcome, when change other things about the utilities, those welfare calculations remain the same. This leads welfare functions to act as IIA preference aggregation rules and Social Choice functions.

1.7 If a welfare function is Monotonic, it is also Pareto Efficient

What would it take to get a welfare function to respect Pareto dominance?

If x strictly Pareto dominates y, we need $W(x) > W(y)$ and if x Pareto dominates y then $W(x) \geq W(y)$.

We call this property **monotonicity.**

A welfare function is **monotonic** if whenever $U_i(x) \geq U_i(y)$ for everyone then $W(x) \geq W(y)$. And if, in addition, there is at least one person for whom $U_i(x) > U_i(y)$ then $W(x) > W(y)$.

In other words, if x Pareto dominates y then $W(x) \geq W(y)$. If x strictly Pareto dominates y then $W(x) > W(y)$.

Using an monotonic welfare function as a preference aggregation rule results in a Pareto efficient preference aggregation rule.

Using an monotonic welfare function as a social choice funciton results in a Pareto efficient social choice function.

So in summary any monotoic welfare function is a complete, transitive, IIA, and Pareto efficient preference aggregation rule. It is also a nonempty, IIA, Pareto efficient social choice function.

1.8 All of the Welfare functions we looked at above are monotonic.

Example: Utilitarian.

Suppose $U_i(x) \geq U_i(y)$ for everyone, then the sum of utilities for x is at least as big as the sum for y. And if one is strict the welfare is strictly larger.