1 Randomization and Side Payments

$O = ab, a, b, n$

$U_a(ab) = 12, U_a(a) = 10, U_a(b) = 25, U_a(n) = 5$

$U_b(ab) = 12, U_b(a) = 25, U_b(b) = 10, U_a(n) = 5$

1.1 Coin-Flip

Suppose that we flip a coin and if it is heads, Alice cleans. If it is tails, Bob cleans.

1.1.1 Expected Utility

An expected utility maximizer is someone who's utility of a "lottery" (a probability distribution over outcomes) is simply assessed by the expected (average) amount of utility the get in that lottery.

$U_a(a) = 10, U_a(b) = 25$

With a coin flip, there is $\frac{1}{2}$ chance of a and $\frac{1}{2}$ chance of b. Her epected utility:

$$E(U_a) = \frac{1}{2}(10) + \frac{1}{2}(25) = 17.5$$

$$E(U_b) = \frac{1}{2}(25) + \frac{1}{2}(10) = 17.5$$

If we pick a different way of randomizing, she gets a different expected utility. $\frac{1}{3}$ of a and $\frac{2}{3}$ b.

$$E(U_a) = \frac{1}{3}(10) + \frac{2}{3}(25) = 20.$$

$$E(U_b) = \frac{1}{3}(25) + \frac{2}{3}(10) = 15.$$

$t = \frac{2}{3}$ (two-thirds chance alice cleans)

$$E(U_a) = \frac{2}{3}(10) + \frac{1}{3}(25) = 15.$$

$$E(U_b) = \frac{2}{3}(25) + \frac{1}{3}(10) = 20.$$

1.2 Convex Combinations

The set of possible expected utilities we can achieve by randomizing between two outcomes, is the set of convex combinations between the two outcomes. (The line between the two outcomes).

Set of possible utilities we can achieve by randomizing between a and b is all the points that can be written this way.

Pick a t between 0 and 1. The probability of a is t and the probability of b is $1-t$.

$$(t\left(10\right)+\left(1-t\right)25, t\left(25\right)+\left(1-t\right)10)$$

For $t=0$

$$(25,10)$$

Suppose we had the follow probability distirbution $s\left(o\right)$ gives the probability of outcome o.

$s\left(a\right)=0.2, s\left(b\right)=0.3, s\left(ab\right)=0.4, s\left(n\right)=0.1$

In general an outcome that is randomized this way is a **randomized outcome**.

$$E\left(U_a\right)=0.2\left(10\right)+0.3\left(25\right)+0.4\left(12\right)+0.1\left(5\right)=14.8$$

$$E\left(U_b\right)=0.2\left(25\right)+0.3\left(10\right)+0.4\left(12\right)+0.1\left(5\right)=13.3$$

1.3 Convex Hull

The convex hull of some points is the set of all points that can be achieved by some randomization over those points.

The smallest convex set that contains all of the original points.

1.4 Achievable Utilities

The set of utilities achievable by some randomized outcome is the convex hull of the utilities achievable in the outcomes themselves.

1.5 Pareto Frontier

ab is no longer Pareto efficient in out cleaning model. For instance, it is strictly Pareto dominated by the coin-flip which gives 17.5 expected utility to both instead of 12.

The Pareto frontier is the set of all of the utility combinations achieveable with randomization that are also Pareto efficient.

Geometrically, it will be the "boundary" of the convex hull of points that lives on the north-east of the hull.

1.6 Maximizing Nash Welfare

1.6.1 Cleaning Kitchen

$O = ab, a, b, n$

$U_a(ab) = 12, U_a(a) = 10, U_a(b) = 25, U_a(n) = 5$

$U_b(ab) = 12, U_b(a) = 25, U_b(b) = 10, U_a(n) = 5$

1.6.2 Maximizing Nash Welfare without Randomization:

$W(ab) = 12^{\frac{1}{2}} 12^{\frac{1}{2}} = 12$

$W(a) = 10^{\frac{1}{2}} 25^{\frac{1}{2}} = 15.8114$

$W(b) = 25^{\frac{1}{2}} 10^{\frac{1}{2}} = 15.8114$

$W(n) = 5^{\frac{1}{2}} 5^{\frac{1}{2}} = 5$

Either a or b maximizes Nash welfare with a welfare value of 15.8114.

Another way of interpreting the amount of welfare is it is the pair of utilities

$$(15.8114, 15.8114)$$

1.6.3 With Randomization?

First off, any welfare function will pick a pareto efficient pair of utility as the winner.

A point that is not on the Pareto frontier will never maximize welfare.

That means, we can focus out effort of finding the optimal point to the Pareto frontier.

For the cleaning model, the Pareto frontier is a line between a and b.

$$(t(10) + (1-t)25, t(25) + (1-t)10)$$

What is the Nash welfare of one of these points?

Take alices utility $t(10) + (1-t)25$ raise it to the power of $\frac{1}{2}$.

Multiply it by

Bob's utility $t(25) + (1-t)10$ raise it to the power of $\frac{1}{2}$.

For any t we choose,

$$\left(t\left(10\right)+\left(1-t\right)25\right)^{\frac{1}{2}}\left(t\left(25\right)+\left(1-t\right)10\right)^{\frac{1}{2}}$$

Let's check that this is true. Pick $t=\frac{1}{2}$

$$\left(\frac{1}{2}\left(10\right)+\left(1-\frac{1}{2}\right)25\right)^{\frac{1}{2}}\left(\frac{1}{2}\left(25\right)+\left(1-\frac{1}{2}\right)10\right)^{\frac{1}{2}} = 17.5$$

Our goal is to maximize welfare. Pick the t that gives the highest welfare.

$$\left(t\left(10\right)+\left(1-t\right)25\right)^{\frac{1}{2}}\left(t\left(25\right)+\left(1-t\right)10\right)^{\frac{1}{2}}$$

An optimization trick is that any function, if you take a strictly increasing function of it, then the resulting function is maximized in the same place as the original.

Let's square it:

$$\left(t\left(10\right)+\left(1-t\right)25\right)\left(t\left(25\right)+\left(1-t\right)10\right)$$

Whatever t maximizes this will also maximize the original function. Let's simplify this:

$$\left(10t+25-25t\right)\left(25t+10-10t\right)$$

$$\left(25-15t\right)\left(15t+10\right)$$

$$-225t^2+225t+250$$

Take the deriviative and find where the slope is zero:

$$\frac{\partial\left(-225t^2+225t+250\right)}{\partial t}=-450t+225=0$$

$$225=450t$$

$$t=\frac{1}{2}$$

1.7 A simple model.

$O = a, b$

$U_1(a) = 25, U_1(b) = 0$

$U_2(a) = 0, U_2(b) = 10$

The set of points on the pareto frontier is the line between a and b. The expected utilities of the convex combinations between these points are:

t is the probability of outcome a

The possible convex combinations are:

$$(t(25) + (1-t)\,0, t(0) + (1-t)\,10)$$

$$(25t, (1-t)\,10)$$

What is the nash welfare of one of these points:

$$(25t)^{\frac{1}{2}}\left((1-t)\,10\right)^{\frac{1}{2}}$$

Since taking this whole thing to the power of 2 is a strictly increasing transformation, the result will be maximized in the same place:

$$(25t)\left((1-t)\,10\right)$$

$$25t\,(10-10t)$$

$$250t - 250t^2$$

To maximize this, find where the derivative is zero:

$$250 - 500t$$

$$t = \frac{1}{2}$$