1 Maximizing Welfare with Randomization

1.1 Running Examples: Fish

$U_a(ys) = 30, U_1(yl) = 30, U_1(ns) = 25, U_1(nl) = 25$

$U_b(ys) = 0, U_2(yl) = 10, U_2(ns) = 20, U_2(nl) = 10$

1.2 Cleaing Example - Nash

Let's take t to be the probability of a. $(10, 25)$

$(1 - t)$ is the probability of b $(25, 10)$.

Alice and Bob's expected utility for t between 0 and 1:

$$(10t + (1 - t)\,25, 25t + (1 - t)\,10)$$

Let's write the welfare (nash) of these points:

$$(10t + (1 - t)\,25)^{\frac{1}{2}}\,(25t + (1 - t)\,10)^{\frac{1}{2}}$$

Our goal is to find the t that maximizes this. Instead of maximizing it directly, maximize it's square.

$$\left((10t + (1 - t)\,25)^{\frac{1}{2}}\,(25t + (1 - t)\,10)^{\frac{1}{2}}\right)^2$$

$$(10t + (1 - t)\,25)\,(25t + (1 - t)\,10)$$

$$(-15t + 25)\,(15t + 10)$$

$$-225t^2 + 225t + 250$$

To maximize this take the derivative set it to zero.

$$-450t + 225 = 0$$

$$t = \frac{1}{2}$$

$\frac{1}{2}$ chance of a and $\frac{1}{2}$ chance of b. This results in expected utilities:

$$(17.5, 17.5)$$

1.3 Fish Example - Nash

The pareto frontier is the line between the points $(30, 10)$ and $(25, 20)$. Let t be the probability of (yl) which gives $(30, 10)$.

t $(30, 10)$

$1 - t$ $(25, 20)$

$$(30t + (1-t)\,25, 10t + (1-t)\,20)$$

The Nash welfare in any of these points is:

$$(30t + (1-t)\,25)^{\frac{1}{2}}\,(10t + (1-t)\,20)^{\frac{1}{2}}$$

Instead of maximizing that, maximize this:

$$(30t + (1-t)\,25)\,(10t + (1-t)\,20)$$

$$-50t^2 - 150t + 500$$

To maximize it, take its derivative and set it to zero:

$$-100t - 150 = 0$$

$$-150 = 100t$$

$$t = -1.5?!?!$$

Since we can't decrease t further than 0, $t = 0$ is the best we can do. So just use the outcome ns which gives $(25, 20)$.

1.4 Maximizing Rawlsian Welfare

If the pareto frontier crosses the $45 - degree$ line, then the point on the pareto frontier and on the 45-degree line (where both people get the same utility) is Rawlsian welfare maximizing.

1.5 Cleaning - Rawls

We can achieve the same amount of utility for both on the Pareto frontier.

$$(10t + (1-t)\,25, 25t + (1-t)\,10)$$

To find the t (the point) that maximizes rawlsian welfare, set the utilities euqal.

$$10t + (1-t)\,25 = 25t + (1-t)\,10$$

Solve for the t that makes them equal.

$$t = \frac{1}{2}$$

$$(17.5, 17.5)$$

1.6 Fish - Rawls

$$30t + (1-t)\,25 = 10t + (1-t)\,20$$

$$t = -\frac{1}{3}$$

Just pick $t = 0$ which is $(25, 20)$ n/s.

1.7 Utilitarian Welfare

One of the endpoints of the pareto frontier will always be a utilitarian maximizing outcome even if randomization is allow. So, there is no real reason to use randomization.

If the utilitarian welfare of one endpoint is strictly higher, that is the utilitarian maximizing point.

If both are the same, and randomization gives the same utilitarian welfare.

2 Side-Payments

With a side payment from (b) $(25, 10)$ we can get to $(17.5, 17.5)$. Have alice pay bob 7.5.

2.1 Maximizing all Three!

First, find the utilitarian welfare maximizing outcome. Calculate the averave utility (utilitarian welfare) of that point. Call that number w. The utility pair (w, w) has the same utilitarian welfare but is perfectly fair. It will maximize all three welfare functions among all achieve outcomes with side-payments.

To find the transfers that get us to this point, simply calculate what payments are needed to get there from the utilitarian welfare maximizing outcome.

2.2 Fish: Side-Payments

The utilitarian welfare maximizing outcome is ns. The utilities are $(25, 20)$which give utilitarian welfare (average utility) of 22.5.

$$(22.5, 22.5) - (25, 20) = (-2.5, 2.5)$$

Alice pays bob $2.5.

2.3 Cleaning: Side-Payments

Either a or b is utilitarian maximizing.

a gives us $(10, 25)$. This has utilitiarian welfare of 17.5.To get to $(17.5, 17.5)$

$$(17.5, 17.5) - (10, 25) = (7.5, -7.5)$$

If we start with b

$$(17.5, 17.5) - (25, 10) = (-7.5, 7.5)$$

alice gives up 7.5 bob gets 7.5.