1 Public Goods

1.1 Two People, Same Preferences

$P \in \{1, 2\}$. Each person chooses a contribution g_1, g_2.

$$u_1 (g_1, g_2) = 20\sqrt{g_1 + g_2} - g_1$$

$$u_2 (g_1, g_2) = 20\sqrt{g_1 + g_2} - g_2$$

$$u_1 (50, 50) = 20\sqrt{50 + 50} - 50 = 150$$

$$u_2 (50, 50) = 20\sqrt{50 + 50} - 50 = 150$$

$$u_1 (100, 0) = 20\sqrt{100 + 0} - 100 = 100$$

$$u_2 (0, 100) = 20\sqrt{100 + 0} - 0 = 200$$

1.2 Best Response

Person 1 has this utility:

$$u_1 (g_1, g_2) = 20\sqrt{g_1 + g_2} - g_1$$

$$\frac{\partial \left(20\sqrt{g_1 + g_2} - g_1\right)}{\partial g_1} = \frac{10}{\sqrt{g_1 + g_2}} - 1$$

Where is this zero?

$$\frac{10}{\sqrt{g_1 + g_2}} = 1$$

$$g_1 + g_2 = 100$$

$$g_1 = 100 - g_2$$

100 is person 1's individually ideal/optimal total contributions.
Person 2 has the same goal:

$$\frac{\partial \left(20\sqrt{g_1 + g_2} - g_2\right)}{\partial g_2} = \frac{10}{\sqrt{g_1 + g_2}} - 1$$

$$g_2 = 100 - g_1$$

In this game, the equilibrium will always involve $g_1 + g_2 = 100$. Define g to be the total contributions $g = g_1 + g_2$. Then in any equilibrium of this game $g = 100$.

As long as $g_1 + g_2 = 100$, the strategies are a Nash equilibrium.

$$(50, 50)$$

$g_1 = 100 - 50 = 50$ 50 from person 1 is a best response to 50 from 2
$g_2 = 100 - 50 = 50$ 50 from person 2 is a best response to 50 from 1

$$(20, 80)$$

$g_1 = 100 - 80 = 20$ 20 from person 1 is a best response to 80
$g_2 = 100 - 20 = 80$, 80 from person 2 is a best response to 20 from 1.

In a public goods model where everyone has the same utility, they all have the same individually ideal total contributes, and any set of contributions that sum to that amount is an equilibrium.

Of all of the combination of strategies that are an equilibrium, $(50, 50)$ is the fairest.

$$u_1(50, 50) = 20\sqrt{50 + 50} - 50 = 150$$

$$u_2(50, 50) = 20\sqrt{50 + 50} - 50 = 150$$

1.2.1 Utilitarian Max

Suppose both contributed 100:

$$u_1(100, 100) = 20\sqrt{100 + 100} - 100.0 = 182.843$$

$$u_2(100, 100) = 20\sqrt{100 + 100} - 100.0 = 182.843$$

What if 1 best responds? Their best response is 0:

$$u_1(0, 100) = 20\sqrt{0 + 100} - 0 = 200$$

They can do even better than 100 each.

What total contribution maximizes utilitarian welfare? What is the ideal size of the park from a utilitarian perspective?

Utilitarian welfare:

$$\frac{20\sqrt{g_1+g_2}-g_1+20\sqrt{g_1+g_2}-g_2}{2}$$

$$\frac{\left(20\sqrt{g_1+g_2}+20\sqrt{g_1+g_2}\right)-(g_1+g_2)}{2}$$

recall that $g = g_1 + g_2$

$$\frac{\left(20\sqrt{g}+20\sqrt{g}\right)-(g)}{2}$$

$$\frac{40\sqrt{g}-g}{2}$$

$$20\sqrt{g}-\frac{1}{2}g$$

$$\frac{\partial\left(20\sqrt{g}-\frac{1}{2}g\right)}{\partial g} = \frac{10}{\sqrt{g}}-\frac{1}{2}$$

$$\frac{10}{\sqrt{g}}-\frac{1}{2}=0$$

$$g = 400$$

The most efficient park size from a utilitarian welfare perspective is 400.
Utilitarian ideal total contributions.

If the players were to agree to a non-equilibrium pair of strategies, picking one where $g_1 + g_2 = 400$ will maximize their average utilities. One way to achieve this is to split up the costs equally and both contribution 200.

$$(200, 200)$$

$$u_1\,(200, 200) = 200$$

$$u_2\,(200, 200) = 200$$

This is the best they can do in this game from utilitarian, rawlsian or Nash welfare.

1.2.2 Tax

One way to achieve this "best" outcome, it to tax each person an equal share of the **utilitarian ideal total contribution.**

$$t = \frac{400}{2} = 200$$

$$u_1(t) = 20\sqrt{t+t} - t$$

$$u_1(200) = 20\sqrt{400} - 200 = 200$$

1.3 Many People, Same Preferences

There are n people: $P \in \{1, 2, .., n\}$

Each chooses their contribution g_i.

Total contributions $g = \sum_{i=1}^{n} g_i$

Total contributions not made by i: $g_{-i} = g - g_i$

Suppose $g_1 = 10, g_2 = 30, g_3 = 40$.

$g = 80$

$g_{-1} = 80 - 10 = 70$

$g_{-2} = 80 - 30 = 50$

$g_{-3} = 80 - 40 = 40$

Utility for i

$$u_i(g_1, g_2, g_3, g_4, ..., g_n) = 20\sqrt{g} - g_i$$

$$u_i(g_i, g_{-i}) = 20\sqrt{g_i + g_{-i}} - g_i$$

To maximize this, find where the derivative with respect to g_i is zero:

$$20\left(g_i + g_{-i}\right)^{\frac{1}{2}} - g_i$$

$$\frac{\partial\left(20\sqrt{g_i + g_{-i}} - g_i\right)}{\partial g_i} = \frac{10}{\sqrt{g_{-i} + g_i}} - 1$$

$$\frac{10}{\sqrt{g_{-i} + g_i}} - 1 = 0$$

$$g_i = 100 - g_{-i}$$

Individually ideal total contribution is 100 for all people. In equilibrium $g = 100$.

1.3.1 Utilitarian Max

Let's look for the g (total contributions that maximize utilitarian welfare).

Utilitarian welfare:

$$\frac{1}{n}\sum_{i=1}^{n}\left(20\sqrt{g_i+g_{-i}}-g_i\right)$$

Let's express this in terms of only g through these steps:

$$\frac{1}{n}\sum_{i=1}^{n}(20\sqrt{g}-g_i)$$

$$\frac{1}{n}\sum_{i=1}^{n}20\sqrt{g}-\frac{1}{n}\sum_{i=1}^{n}g_i$$

$$\frac{1}{n}n\,(20\sqrt{g})-\frac{1}{n}g$$

$$20\sqrt{g}-\frac{1}{n}g$$

Where is this maximized?

$$\frac{\partial\left(20\sqrt{g}-\frac{1}{n}g\right)}{\partial g} = \frac{10}{\sqrt{g}}-\frac{1}{n}$$

$$\frac{10}{\sqrt{g}}-\frac{1}{n}=0$$

$$10n=\sqrt{g}$$

$$g=100n^2$$

1.4 Case of $n=1000$

In equilibrium the total contributions are: 100.

The utilitarian ideal:

$$g=100\,(1000)^2=100000000$$

One way to raise this amount of money is to use the utilitarian ideal tax:

$$t = \frac{100000000}{1000} = 100000$$

Utility of this:

$$20\sqrt{100000000} - 100000 \quad = \quad 100000$$

In equilibrium:

$$20\sqrt{100} - \frac{100}{1000} \quad = \quad 199.9$$

1.5 Individually Ideal Tax.

Suppose the government decides to levy a tax. As any person what tax the government should charge? What is their favorite tax?

$$u_i\left(t\right) = 20\sqrt{n * t} - t$$

What t maximizes this:

$$\frac{\partial\left(20\sqrt{n * t} - t\right)}{\partial t} \quad = \quad \frac{10n}{\sqrt{nt}} - 1$$

$$\frac{10n}{\sqrt{nt}} - 1 = 0$$

$$t = 100n$$

$n = 1000$ then $t = 100000$
This is exactly the utilitarian ideal.