1 Public Goods Continued

$$u_i\left(g_i, g_{-i}\right) = a_i\sqrt{g_i + g_{-i}} - g_i$$

$$u_i\left(g_i, g_{-i}\right) = a_i\sqrt{g} - g_i$$

$a_1 = 10, a_2 = 20, a_3 = 60$

1.1 Equilibrium

Person i wants to maximize their utility by choosing g_i:

Where is the slope of their utility equal to zero?

$$\frac{\partial\left(a_i\sqrt{g_i + g_{-i}} - g_i\right)}{\partial g_i} = \frac{a_i}{2\sqrt{g_{-i} + g_i}} - 1$$

$$\frac{a_i}{2\sqrt{g_{-i} + g_i}} - 1 = 0$$

$$\frac{a_i}{2\sqrt{g_{-i} + g_i}} = 1$$

$$a_i = 2\sqrt{g_{-i} + g_i}$$

$$\frac{1}{2}a_i = \sqrt{g_{-i} + g_i}$$

$$\frac{1}{4}a_i^2 = g_{-i} + g_i$$

$$g_i = \frac{1}{4}a_i^2 - g_{-i}$$

For person 1, $a_1 = 10$

$$g_1 = 25 - g_{-i}$$

So, for example, if person 2 and 3 contribute a total of 10 then person 1's best response is $g_1 = 25 - 10 = 15$.

$$g_2 = 100 - g_{-2}$$

$$g_3 = 900 - g_{-3}$$

In equilibrium of a public goods problem with heterogeneous (different) preferences, the highest individually ideal total contributes (here, 900) is the equilibrium total contributes and the only people who contribute anything are those with the highest individually ideal total contributions.

$$(0, 0, 900)$$

1.2 Utilitarian Optimal

We can calculate the efficient (maximizing utilitarian welfare) level of total contributes.

First write the average utility for any set of contributions:

$$\frac{\left(10\sqrt{g} - g_1\right) + \left(20\sqrt{g} - g_2\right) + \left(60\sqrt{g} - g_3\right)}{3}$$

$$\frac{10\sqrt{g} + 20\sqrt{g} + 60\sqrt{g} - g_1 - g_2 - g_3}{3}$$

$$\frac{90\sqrt{g} - (g_1 + g_2 + g_3)}{3}$$

$$\frac{90\sqrt{g} - g}{3}$$

What g maximizes this?

Look for where the slope is zero:

$$\frac{\partial\left(\frac{90\sqrt{g}-g}{3}\right)}{\partial g}=0$$

$$\frac{1}{3}\left(\frac{45}{\sqrt{g}}-1\right)=0$$

$$\frac{45}{\sqrt{g}}=1$$

$$45=\sqrt{g}$$

$$g=2025$$

Utilitarian Ideal (Optimal) Total Contributions

1.3 Individually Ideal Taxes

One way to achieve this is to charge each person a equal amount of the total contributions.

Utilitarian ideal taxes is $t=\frac{2025}{3}$. $t=675$

1.4 How Hard it Incentive Compatibility

Let's ask each person what their a_i is.

We want person 1 to say $a_1=10$.

If everyone tells the truth, we can use this information to implement the utilitarian optimal tax.

Suppose they say their preferences are a_1, a_2, a_3:

$$\frac{a_1\sqrt{g}+a_2\sqrt{g}+a_3\sqrt{g}-g}{3}$$

$$\frac{(a_1+a_2+a_3)\sqrt{g}-g}{3}$$

$$\frac{\partial\left(\frac{(a_1+a_2+a_3)\sqrt{g}-g}{3}\right)}{\partial g} = \frac{1}{3}\left(\frac{a_1+a_2+a_3}{2\sqrt{g}}-1\right)$$

$$\frac{1}{3}\left(\frac{a_1+a_2+a_3}{2\sqrt{g}}-1\right)=0$$

$$g=\frac{1}{4}\left(a_1+a_2+a_3\right)^2$$

The tax charged

$$t=\frac{\frac{1}{4}\left(a_1+a_2+a_3\right)^2}{3}$$

Under this "mechanism" the utility of each person is:

$$u_i=a_i\sqrt{\frac{1}{4}\left(a_1+a_2+a_3\right)^2}-\frac{\frac{1}{4}\left(a_1+a_2+a_3\right)^2}{3}$$

Let's look at the incentives of person 3 when person 1 and 2 tell the truth:

$$u_3=60\sqrt{\frac{1}{4}\left(10+20+a_3\right)^2}-\frac{\frac{1}{4}\left(10+20+a_3\right)^2}{3}$$

Let's suppose they tell the truth:

$a_3=60$ and they get utility $u_3=2025$

What if they lie? What is the best thing they could say?

It is the a_3 that maximizes this:

$$a_3=150$$

This is not an incentive compatible mechanism.

1.5 Median Mechanism

We can't implement the utilitarian optimal tax without knowing the individual preferences, but can we implement?

We ask each person what their favorite tax is. Then we arrange the taxes in ascending order, and pick the on in the middle- this is the **median voter's favorite tax.**

This tax is the only tax t that would win in a pairwise vote against any other tax. It is the only **condorcet** winner.

If you pick any other tax, then a majority (50+%) would prefere the median voters favorite tax instead.

It is Pareto efficient among all policies that charge everyone the same tax.

Let's look at our example.

We need to calculate everyone's individually optimal tax:

$$a_i\sqrt{3*t}-t$$

We can calculate the indiviudally tax by maximizing this:

$$\frac{\partial\left(a_i\sqrt{3*t}-t\right)}{\partial t} = \frac{\sqrt{3}a_i}{2\sqrt{t}}-1$$

$$\frac{\sqrt{3}a_i}{2\sqrt{t}}-1=0$$

$$t=\frac{3a_i^2}{4}$$

For our case the individually ideal taxes are respectively:

$$75, 300, 2700$$

The median voter's favorite tax is 300.

Despite the fact that there was no incentive for truth-telling when we used the preference information to calculate the utilitarian optimal tax, here everyone has incentive to tell the truth. It is incentive-compatible.

This is a decisive and non-manipulable mechanism.

1.6 Pareto Dominating Median Mechanism

We know $t = 300$ is Pareto efficient. It is not Pareto dominated by any policy that has everyone pay the same tax.

Let's caluclate the utility of each person under $t = 300$:

$$a_i\sqrt{900} - 300$$

These are their respective utilities:

$$0, 300, 1500$$

Even compared to the utilitarian ideal $t = 675$.

$$\{-225, 225, 2025\}$$

However, even though 300 is Pareto dominated by any policy where everyone pays the same tax, it is dominated by a policy that uses individualized taxes. (Different people pay different amounts).

Total utility under $t = 300$ is $0 + 300 + 1500 = 1800$

Total utility under $t = 675$ is $-225 + 225 + 2025 = 2025$

$$2025 - 1800 = 225$$

Let's try to give each person $\frac{225}{3} = 75$ more then they got under $t = 300$.

These are their utilities under the median mechanism:

$$0, 300, 1500$$

Can we give each person 75 more?

$$75, 375, 1575$$

We know to achieve the total contributions need to be $675 * 3 = 2025$.

If we can get person 1 to have a utility of 75:

$$10\sqrt{2025} - t_1 = 75$$

$$t_1 = 375$$

$$20\sqrt{2025} - t_2 = 375$$

$$t_2 = 525$$

$$60\sqrt{2025} - t_3 = 1575$$

$$t_3 = 1125$$

Median mechanism chooses $t = 300$ for everyone.

If instead, we charge $t_1 = 375, t_2 = 525, t_3 = 1125$ then everyone is 75 better off then are under the median mechanism.