1 Exercises

1.1 11.1-11.3

Calculate Nash welfare for the four points.

$W(u_a, u_b) = (u_a)^{\frac{1}{2}} (u_b)^{\frac{1}{2}}$

$$W(25,5) = 25^{\frac{1}{2}} 5.0^{\frac{1}{2}} = (25 * 5)^{\frac{1}{2}} = \sqrt{125}$$

$$W(25,10) = 25^{\frac{1}{2}} 10^{\frac{1}{2}} = (25 * 10)^{\frac{1}{2}} = \sqrt{250}$$

$$W(10,20) = 10^{\frac{1}{2}} 20^{\frac{1}{2}} = (20 * 10)^{\frac{1}{2}} = \sqrt{200}$$

$$W(10,10) = 10^{\frac{1}{2}} 10^{\frac{1}{2}} = (10 * 10)^{\frac{1}{2}} = \sqrt{100}$$

The Pareto Frontier is the line between the points $(25,10), (10,20)$.

Pick a t between 0 and 1. Let t be the probability of $(25,10)$.

$$(25t + (1-t)10, 10t + (1-t)20)$$

The Nash welfare of any of these points is:

$$(25t + (1-t)10)^{\frac{1}{2}} (10t + (1-t)20)^{\frac{1}{2}}$$

Take a transformation of this by squaring the whole thing:

$$(25t + (1-t)10)(10t + (1-t)20)$$

$$-150t^2 + 200t + 200$$

Where is the slope of this zero?

$$\frac{\partial\left(-150t^2 + 200t + 200\right)}{\partial t} = 200 - 300t$$

$$200 = 300t$$

$$t = \frac{2}{3}$$

$\frac{2}{3}$ of $(25,10)$ and $\frac{1}{3}$ of $(10,20)$

2 Public Goods

2.1 Cleaning Game

s_a/s_b	bob:y	bob:n
alice:y	12,12	**10**,**35**
alice:n	**35**,**10**	5,5

Equilibrium - singular

Equilibria- plural

The Nash equilibria are (n, y) and (y, n)

2.2 Coffee Machine Game

	bob:y	bob:n
alice:y	$15, 15$	5,**20**
alice:n	**20**,5	**10**,**10**

2.3 Continuous Strategies

People/Players: $P = \{1, 2\}$

Each person picks g_i which is the amount of money they contribute to the park.

Strategies: $g_1 \in [0, \infty)$, $g_2 = [0, \infty)$

Preferences/Payoffs/Utilities

2.3.1 Utility Function

Let's suppose each person's utility from money contributed to the park is:

$$10 \log (g_1 + g_2)$$

$$u_1 (g_1, g_2) = 10 \log (g_1 + g_2) - g_1$$

$$u_2 (g_1, g_2) = 10 \log (g_1 + g_2) - g_2$$

Let g be the sum of contributions: $g = g_1 + g_2$

2.3.2 Best Response Function

What is the optimal amount of contribution g_1 if person 2 contributes g_2? $B_1(g_2)$?

Have Alice maximize her utility subject to Bob picking g_2. That happens where the slope of her utility function with g_1 is zero.

$$\frac{\partial\left(10\log\left(g_1+g_2\right)-g_1\right)}{\partial g_1} = \frac{10}{g_1+g_2}-1$$

Where is this zero H

$$\frac{10}{g_1+g_2}-1=0$$

$$10=g_1+g_2$$

Person 1's best response is to get the total contributions to 10:

$$g_1=10-g_2$$

$B_1(g_2)=10-g_2$

$$u_2\left(g_1,g_2\right)=10\log\left(g_1+g_2\right)-g_2$$

$$\frac{\partial\left(10\log\left(g_1+g_2\right)-g_2\right)}{\partial g_2} = \frac{10}{g_1+g_2}-1$$

$$g_2=10-g_1$$

$B_2(g_1)=10-g_1$

2.3.3 Equilibrium

A pair (g_1,g_2) is a Nash equilibrium if $B_1(g_2)=g_1$ and $B_2(g_1)=g_2$

Any g_1 and g_2 that sum to 10 is Nash equilibrium.

Here are there best responses:

$$g_1 = 10 - g_2$$

$$g_2 = 10 - g_1$$

$(8, 2)$,

$$g_1 = 10 - 2$$

$$g_1 = 8$$

$$g_2 = 10 - 8$$

$$g_2 = 2$$

$(5, 5)$ is also a Nash equilibrium. In the Nash equilibrium, whichever one it is, the total contributions are 10.

$$u_1 (5, 5) = 10 \log (5 + 5) - 5.0$$

$$(18.0259, 18.0259)$$

2.3.4 Utilitarian Max

What contributions maximize total utility? What is the utilitarian maximizing pairs of g_1, g_2

Let's sum their utilities to get the utilitarian welfare:

$$10 \log (g_1 + g_2) - g_1 + 10 \log (g_1 + g_2) - g_2$$

$$2 * 10 \log (g_1 + g_2) - g_1 - g_2$$

$$2 * 10 \log (g) - g$$

Where is this maximized:

$$\frac{\partial \left(2 * 10 \log \left(g\right) - g\right)}{\partial g} = \frac{20}{g} - 1$$

Where is this zero?

$$\frac{20}{g} = 1$$

$$g = 20$$

Utilitarian welfare is maximized where total contributions are 20. For instance $(10, 10)$.

$$u\left(10, 10\right) = 19.9573$$

2.3.5 Taxation

The government can make both people better off by mandating that they give 10 to the park. They can do this by taxing each person $t = 10$ and using those to build the park.

2.3.6 Favorite tax

Suppose we ask each individual to pick a contribution that both people have to make:

$$10 \log \left(g_1 + g_2\right) - g_1$$

Pick a t (a tax) that both people have to give. The policy t creates this utility:

$$10 \log \left(t + t\right) - t$$

Each persons favorite tax is the one that maximizes their utility:

$$\frac{\partial \left(10 \log \left(t + t\right) - t\right)}{\partial t} = \frac{10}{t} - 1$$

$$\frac{10}{t} - 1 = 0$$

$$10 = t$$