1 Chapter 10

1.1 Plotting Outcomes

$u_1(a) = 5, u_1(b) = 15, u_1(c) = 8$

$u_2(a) = 25, u_2(b) = 5, u_2(c) = 8$

1.2 The Three Welfare Functions

1.2.1 Utilitarian

"Average utility"

$W(a) = \frac{5+25}{2} = 15$

$W(b) = \frac{15+5}{2} = 10$

$W(c) = \frac{8+8}{2} = 8$

a maximizes utilitarian welfare.

1.2.2 Rawlsian

"Min Utility"

$W(a) = 5$

$W(b) = 5$

$W(c) = 8$

c maximizes rawlsian welfare.

1.2.3 Nash

"Product of the square roots"

$u_a^{\frac{1}{2}} u_b^{\frac{1}{2}}$

$u_1(a) = 5, u_1(b) = 15, u_1(c) = 8$

$u_2(a) = 25, u_2(b) = 5, u_2(c) = 8$

$W(a) = \sqrt{5}\sqrt{25} = \sqrt{5 * 25} = \sqrt{125}$

$W(b) = \sqrt{15}\sqrt{5} = \sqrt{15 * 5} = \sqrt{75}$

$W(c) = \sqrt{8}\sqrt{8} = \sqrt{64}$

Which maximizes Nash welfare? a

2 Chapter 11

2.1 Find all points achievable with randomization.

Take the convex hull of the utility pairs achievable with non-randomized outcomes.

2.2 Find the Pareto frontier

The set of all points achievable with randomization that are Pareto efficient.

2.3 Writing a Convex Combination

A convex combination is the set of all points "between" two others.

$u_1(a) = 5, u_1(b) = 15, u_1(c) = 8$

$u_2(a) = 25, u_2(b) = 5, u_2(c) = 8$

a is $(5, 25)$ b is $(15, 5)$

What are all the points between a and b?

t is the probability we pick a. $(1 - t)$ is the probability we pick b

What is the expected (average) utility for person 1?

$$t(5) + (1 - t) 15$$

$$15 - 10t$$

What is the expected (average) utility for person 2?

$$t(25) + (1 - t) 5$$

$$20t + 5$$

The points achieveable on the Pareto frontier:

$$(15 - 10t, 20t + 5)$$

For $t = \frac{1}{2}$

$$\left(15 - 10\frac{1}{2}, 20\frac{1}{2} + 5\right)$$

$$(10, 15)$$

2.4 Find points on frontier that maximize welfare.

Of the points on a PF, which maximizes:

2.4.1 Utilitarian

For utilitarian at least one of the endpoints will be a maximum.

Check the end-points, the outcome with highest utilitarian is the maximum even among possible randomized outcomes.

$W(a) = \frac{5+25}{2} = 15$

$W(b) = \frac{15+5}{2} = 10$

Since a has a higher utilitarian welfare, it is the maximum on the Pareto frontier.

If both had been the same, any point on the Pareto frontier will maximize utilitarian welfare.

Rawlsian If there is point on the Pareto frontier where they get the same utility, that maximizes Rawlsian welfare.

$$(15 - 10t, 20t + 5)$$

If we set the utilities equal:

$$15 - 10t = 20t + 5$$

Solve for t :

$$10 = 30t$$

$$t = \frac{1}{3}$$

Pick a with probability $\frac{1}{3}$ and b with probability $\frac{2}{3}$

$$(11.6667, 11.6667)$$

If we were to find a t either bigger than 1 or smaller than 0

If this happens, then whichever of the endpoints has higher rawlsian welfare, that is the point on the Pareto frontier that maximizes rawlsian welfare.

2.4.2 Nash

Of all of the points: $(15 - 10t, 20t + 5)$ which maximizes Nash welfare?

What is the Nash welfare for any of these points?

$$\sqrt{15 - 10t}\sqrt{20t + 5}$$

This is a one dimensional function. To maximize it, look for where the derivative is zero.

The trick is to maximize this instead:

$$(15 - 10t)\,(20t + 5)$$

This is a quadratic function:

$$-200t^2 + 250t + 75$$

This is maximized where the derivative is zero:

$$\frac{\partial\left(-200t^2 + 250t + 75\right)}{\partial t} = -400t + 250$$

$$-400t + 250 = 0$$

$$400t = 250$$

$$t = \frac{250}{400} = \frac{5}{8}$$

2.5 Find set of points achievable with side-payments

Draw a line with slope of -1 through the point with higest utilitarian welfare.

2.6 Find point with side payments that maximizes welfare.

The point that simultaneously maximizes all welfare functions when side payments are possible is the the point on the line where both utilities are the same.

A quick way to find this point is to look at the utilitarian welfare of the outcome with the highest utilitarian welfare.

$W(a) = \frac{5+25}{2} = 15$

With side payments, we can give both people the average utility that they get in the best utilitarian outcome.

$$(15, 15)$$

To get this point, start with outcome a $(5, 25)$ have person 2 give person 1 \$10.

3 Chapter 12

s_1, s_2	a	b
a	$10, 10$	$5, 20$
b	$20, 5$	$0, 0$

3.1 Find the Pareto Efficient Outcomes of a Game

Strategies (a, a) leads to utility pair $(10, 10)$

Strategies (a, b) leads to utility pair $(5, 20)$

Strategies (b, a) leads to utility pair $(20, 5)$

Strategies (b, b) leads to utility pair $(0, 0)$

Here $(a, a), (a, b), (b, a)$ are the Pareto efficient outcomes.

3.2 Find the Best Responses in a Game

A best response is a strategy that maximizes utility given what the other's strategy is.

s_1, s_2	a	b
a	$10, 10$	$\mathbf{5, 20}$
b	$\mathbf{20, 5}$	$0, 0$

The best response for 1 if 2 plays 'a':

$B_1(a) = b, B_1(b) = a$

$B_2(a) = b, B_2(b) = a$

3.3 Find the Nash equilibrium in a Game

A Nash equilibrium are a set of strategies that mutual best responses.

(a, a) a Nash equilibrium? No since a for either 1 or 2 is not a best response to a

(b, b) not a Nash equililibrium.

(a, b) is a Nash equilibrium

(b, a) is a Nash equilibrium

These are the cells in the above game where both numbers are highlighted.

4 Chapter 13/14

Two utility functions $a_i \ln(g_i + g_{-i}) - g_i$ and $a_i\sqrt{g_i + g_{-i}} - g_i$

4.1 Find Best responses in a public goods game.

3 people have utility:

$$a_i \ln(g_i + g_{-i}) - g_i$$

$a_1 = 100, a_2 = 200, a_3 = 500$

To find the best response, maximize i's utility with respect g_i. Find where the derivative of utility with respect to g_i is zero:

$$\frac{\partial(a_i \ln(g_i + g_{-i}) - g_i)}{\partial g_i} = 0$$

$$\frac{a_i}{g_i + g_{-i}} - 1 = 0$$

Solve this for g_i to get the best response:

$$\frac{a_i}{g_i + g_{-i}} - 1 = 0$$

$$a_i = g_i + g_{-i}$$

$$g_i = a_i - g_{-i}$$

Three best responses are:

$$g_1 = 100 - g_{-1}$$

$$g_2 = 200 - g_{-2}$$

$$g_3 = 500 - g_{-3}$$

4.2 Find individually ideal total contributions.

$100, 200, 500$ respectively

4.3 Find equilibrium total contributions

When people have different preferences, in equilibrium only the person with the highest individually ideal total contribution will contribute anything and they will contribute that individually ideal total themselves.

$$(0, 0, 500)$$

If we have everyone with the same preferences

$a_1 = 100, a_2 = 100, a_3 = 100$

Individually ideal totals $100, 100, 100$

Any contributions that sum to 100 is an equilibrium.

$$(25, 25, 50)$$

4.4 Total contributions that maximize utilitarian welfare.

$$\frac{100\ln(g) - g_1 + 200\ln(g) - g_2 + 500\ln(g) - g_3}{3}$$

$$\frac{800}{3} ln(g) - \frac{1}{3} g$$

What g maximizes this?

$$\frac{800}{3}\frac{1}{g} - \frac{1}{3} = 0$$

$$\frac{800}{3}\frac{1}{g} = \frac{1}{3}$$

$$800 = g$$

4.5 Find utilitarian ideal tax

Split up the utilitarian maximizing total contributions equally:

$$t = \frac{800}{3}$$

4.6 Find individually ideal taxes

Under a policy where everyone is taxed t each has a utility:

$$a_i \ln(3t) - t$$

$$\frac{\partial\left(a_i \ln(3t) - t\right)}{\partial t} = 0$$

$$t \to a_i$$

$(100, 200, 500)$ individually ideal taxes

4.7 Find the median voter's favorite tax

$t = 200$

5 Chapter 15

$c = 1000$

$v_1 = 700, v_2 = 400, v_3 = 200$

If is efficient to purchase the TV? What is the efficient outcome?

$\sum v_i \geq c$ then it is efficient to produce/buy the public good.

It is efficient to buy the TV since $v_1 + v_2 + v_3 = 1300 > 1000$.

5.1 Unanimous Agreement Mechanism

If everyone's valuation is above $v_i \geq \frac{c}{n}$ then the public good is provided and each person pays $t_i = \frac{c}{n}$.

In this case, they do not buy the public good and no one pays anything.

Is this the efficient decision? No. The efficient decision is to buy the TV.

5.1.1 another example

$c = 1000$

$v_1 = 700, v_2 = 400, v_3 = 350$

Here because everyone's valuation is above $\frac{1000}{3}$, they buy the TV and each person pays $\frac{1000}{3} = 333.33..$

5.2 Median Mechanism

If the median $v_i \geq \frac{c}{n}$ then the public good is provided and each person pays $\frac{c}{n}$.

$v_1 = 700, v_2 = 400, v_3 = 200$

Since $400 > \frac{1000}{3}$ they buy the TV and each person pays $\frac{1000}{3}$.

This is the efficient decision.

5.2.1 Another Example

$c = 1000$

$v_1 = 400, v_2 = 400, v_3 = 100$

$v_1 + v_2 + v_3 = 900,\ c = 1000.$

The median mechanism leads to them buying the TV because $400 \geq \frac{1000}{3}$. Each person pays $\frac{1000}{3}$.

5.2.2 Yet Another Example

$c = 1000$

$v_1 = 2000, v_2 = 100, v_3 = 100$

$v_1 + v_2 + v_3 = 2200,\ c = 1000.$

The median mechanism: they don't buy the TV because $100 < \frac{1000}{3}$.

This is inefficient since they should buy they TV.

5.3 VCG Mechanism

The VCG mechanism always picks the efficient outcome.

In the VCG mechanism only **pivotal** people pay.

Pivotal means your existence changes whether it is efficient to produce the public good.

$c = 1000$

$v_1 = 700, v_2 = 400, v_3 = 200$

It is efficient to produce the public good (buy the TV)

Who is pivotal?

Person 1 is pivotal since if they didn't exists the total valuation would be $600 < 1000$.

Person 2 is pivotal since if they didn't exists the total valuation would be $900 < 1000$.

Person 3 is **not** pivotal since if they didn't exists the total valuation would be $1100 > 1000$ (it is still efficient to buy the tv).

Transfers they pay are:

Suppose $v_{-i} = \left(\sum_{j=1}^{n} v_j\right) - v_i$

$v_1 = 700, v_2 = 400, v_3 = 200$

For example $v_{-1} = 400 + 200 = 600$

$v_{-2} = 700 + 200 = 900$

$v_{-3} = 700 + 400 = 1100$

The amount someone who is pivotal pays is:

$$t_i = c - (v_{-i})$$

$$t_1 = 1000 - 600 = 400$$

$$t_2 = 1000 - 900 = 100$$

$$t_3 = 0$$

Is there a budget deficit?

Yes, they only collect 500 but the TV costs 1000.

6 More Public Good Problems

Each person chooses how much to contribute to a public good: g_i

g is the total contributions $\sum_{i=1}^{n} g_i = g$

g_{-i} is the total contributions except for i. $g_{-i} = g - g_i$

$$a_i\sqrt{g} - g_i = a_i\sqrt{g_i + g_{-i}} - g_i$$

Suppose we have 3 people with $a_i = 100$.

What is the best response of person i. That is, what is the optimal contribution for person i given the contributions of other g_{-i}?

What g_i maximizes i's utility?

$$a_i\sqrt{g_i + g_{-i}} - g_i$$

We solve:

$$\frac{\partial\left(a_i\sqrt{g_i + g_{-i}} - g_i\right)}{\partial g_i} = 0$$

$$a_i\frac{1}{2}\left(g_i + g_{-i}\right)^{-\frac{1}{2}}(1) - 1 = 0$$

$$\frac{1}{2}a_i\frac{1}{\sqrt{g_i + g_{-i}}} - 1 = 0$$

$$\frac{1}{2}a_i\frac{1}{\sqrt{g_i + g_{-i}}} = 1$$

$$\frac{1}{2}a_i = \sqrt{g_i + g_{-i}}$$

$$\frac{1}{4}a_i^2 = g_i + g_{-i}$$

$$g_i = \frac{1}{4}a_i^2 - g_{-i}$$

$a_i = 100$

$$g_i = 2500 - g_{-i}$$

$a_1 = 100, a_2 = 100, a_3 = 100$

Individually ideal total contributions for each person is 2500.

What the equilibrium of this game?

Any contributions that sum to 2500 is an equilibrium.

$$(1000, 500, 1000)$$

6.0.1 Heterogeneous Example

$a_1 = 100, a_2 = 200, a_3 = 400$

What are the individually ideal total contributions for each person?

$$g_i = \frac{1}{4}a_i^2 - g_{-i}$$

$$g_1 = 2500 - g_{-1}$$

$$g_2 = 10000 - g_{-2}$$

$$g_3 = 40000 - g_{-3}$$

The individually ideal total contributions are $2500, 10000, 40000$ respectively.

The only equilibrium is that the person who cares most contributions and they contribute their entire ideal total contributions.

$$(0, 0, 40000)$$

6.0.2 Utilitarian Efficient Contributions

What total contributions g maximize utilitarian welfare?

You'll always be able to write down utilitarian welfare just in terms of g.

$$\frac{1}{3}\sum_{i=1}^{3}\left(a_i\sqrt{g}-g_i\right)=\frac{1}{3}\left(\sum_{i=1}^{3}a_i\sqrt{g}-\sum_{i=1}^{3}g_i\right)$$

$$=\frac{1}{3}\left(\sum_{i=1}^{3}a_i\sqrt{g}-g\right)$$

$$=\frac{\left(a_1+a_2+a_3\right)\sqrt{g}-g}{3}$$

Where is this maximized:

$$\frac{\partial\frac{\left(a_1+a_2+a_3\right)\sqrt{g}-g}{3}}{\partial g}=0$$

$a_1=100, a_2=200, a_3=400$

$$\frac{\partial\left(\frac{700}{3}\sqrt{g}-\frac{1}{3}g\right)}{\partial g} = \frac{350}{3\sqrt{g}}-\frac{1}{3}$$

$$\frac{350}{3\sqrt{g}}-\frac{1}{3}=0$$

$$\frac{350}{\sqrt{g}}=1$$

$$\sqrt{g}=350$$

$$g=122500$$

To achieve this total contribution with a tax we can use the "utilitarian ideal tax".

$$\frac{122500}{3}=40833.3$$

6.0.3 Individually Ideal Taxes

If each person could choose what tax t is charged to everyone, what is the favorite tax for each person i? **Individually ideal tax**.

Person i's utility under tax t is:

$$a_i\sqrt{3t} - t$$

Maximize this to find i's favorite tax:

$$\frac{\partial \left(a_i\sqrt{3t} - t\right)}{\partial t} = 0$$

$$\frac{\partial \left(a_i \left(3t\right)^{\frac{1}{2}} - t\right)}{\partial t}$$

$$\frac{1}{2}a_i\frac{1}{\sqrt{3t}}3 - 1 = 0$$

$$\frac{3}{2}a_i = \sqrt{3t}$$

$$\frac{9}{4}a_i^2 = 3t$$

$$\frac{3}{4}a_i^2 = t$$

The individually ideal taxes are:

$$\frac{3}{4}\left(100\right)^2, \frac{3}{4}\left(200\right)^2, \frac{3}{4}\left(400\right)^2$$

$$7500, 30000, 120000$$

What is the median voter's favorite tax:

$$30000$$

6.1 Log Utility Utilitarian Welfare

For log utility the utilitarian welfare would be:

$$= \frac{(a_1 + a_2 + a_3)\, ln\, (g) - g}{3}$$

6.2