1 Chapter 10

1.1 Plotting Outcomes

$u_1(a) = 5, u_1(b) = 15, u_1(c) = 8$

$u_2(a) = 25, u_2(b) = 5, u_2(c) = 8$

$a : (5, 25), b : (15, 5), c : (8, 8)$

1.2 The Three Welfare Functions

1.2.1 Utilitarian

Average Utility.

$$\frac{u_a + u_b}{2}$$

$a : (5, 25), b : (15, 5), c : (8, 8)$

$W(a) = \frac{5+25}{2} = 15, W(b) = \frac{15+5}{2} = 10, W(c) = \frac{8+8}{2} = 8$

a maximizes rawlsian welfare

1.2.2 Rawlsian

Minimum Utility

$$min\{u_a, u_b\}$$

$a : (5, 25), b : (15, 5), c : (8, 8)$

$W(a) = 5, W(b) = 5, W(c) = 8$

c maximizes rawlsian welfare.

1.2.3 Nash

$\sqrt{u_a}\sqrt{u_b}$

$a : (5, 25), b : (15, 5), c : (8, 8)$

$W(a) = \sqrt{5}\sqrt{25} = \sqrt{25 * 5} = \sqrt{125}$

$W(b) = \sqrt{15}\sqrt{5} = \sqrt{75}$

$W(c) = \sqrt{8}\sqrt{8} = \sqrt{64}$

a maximizes Nash welfare

2 Chapter 11

2.1 Find all points achievable with randomization.

Find the convex hull of the points.

2.2 Find the Pareto frontier

The pareto frontier are the points in the convex hull that are Pareto efficient.

Find the edge on the North-east of the convex hull.

2.3 Writing a Convex Combination

We need to be able to write down a "formula" for all the points on the Pareto frontier.

The Pareto frontier is an edge of the convex. That is, it is a line between two of the outcomes.

Here the Pareto frontier is the line between a and b.

To mathematically represent a line between points, we use a "convex combination".

What is the expected utility of person 1 if we pick a with probability t and b with $(1-t)$?

$$t5 + (1-t)\,15$$

$$15 - 10t$$

What is the expected utility of person 2 if we pick a with probability t and b with $(1-t)$?

$$t25 + (1-t)\,5$$

$$20t + 5$$

What are all the points we can achieve by randomizing betwen a and b (all the points on the Pareto frontier:

$$(15 - 10t, 20t + 5)$$

2.4 Find points on frontier that maximize welfare.

2.4.1 Utilitarian

You can never do better than one of the endpoints.

Whatever the best endpoint is maximizes utilitarian welfare on Pareto frontier.

$W(a) = 15, W(b) = 10$

a

2.4.2 Rawlsian

To maximize Rawlsian welfare, we look for the point where both utilities are the same.

What point on the pareto frontier has the feature that $u_1 = u_2$

$$(15 - 10t, 20t + 5)$$

$$15 - 10t = 20t + 5$$

$$30t = 10$$

$$t = \frac{1}{3}$$

$$\left(15 - 10\frac{1}{3}, 20\frac{1}{3} + 5\right)$$

$$\left(\frac{35}{3}, \frac{35}{3}\right)$$

This is achieved by picking a with probability $\frac{1}{3}$ and b with probability $\frac{2}{3}$

2.4.3 Nash

We want to maximize the product of the square roots of the utilities:

$$\sqrt{15 - 10t}\sqrt{20t + 5}$$

A trick we can use is to instead maximize this:

$$(15 - 10t)(20t + 5)$$

$$-200t^2 + 250t + 75$$

$$\frac{\partial\left(-200t^2 + 250t + 75\right)}{\partial t} = -400t + 250$$

$$-400t + 250 = 0$$

$$t = \frac{5}{8}$$

2.5 Find set of points achievable with side-payments

2.6 Find point with side payments that maximizes welfare.

The point that gives each person the utilitarian welfare of the outcome that has the best utilitarian welfare simultaneously maximizes all welfare functions among all points achevable with side-payments.

Since $W(a) = 15$ we give both 15. by choosing outcome a which gives $(5, 25)$ and have person 2 give up 10 to person 1.

3 Chapter 12

s_1, s_2	a	b
a	$10, 10$	$5, 20$
b	$20, 5$	$0, 0$

3.1 Find the Pareto Efficient Outcomes of a Game

$a, a : (10, 10)\,; a, b : (5, 20)\,; b, a : (20, 5)\,; b, b : (0, 0)$

$a, a; a, b; b, a$

3.2 Find the Best Responses in a Game

The best response is like a strategy guide for a player.

s_1, s_2	a	b
a	$10, 10$	$\mathbf{5, 20}$
b	$\mathbf{20, 5}$	$0, 0$

What should 1 choose if 2 chooses a? $B_1(a) = b, B_1(b) = a$

$B_2(a) = b, B_2(b) = a$

3.3 Find the Nash equilibrium in a Game

Is a set of mutual best responses. That is, a strategy for 1 that is a best response to 2's strategy and vise versa.

$(a, b), (b, a)$

3.4 Another Nash Example

s_1, s_2	a	b
a	$15, 15$	$0, 20$
b	$\mathbf{20}, 0$	$\mathbf{5, 5}$

4 Chapter 13/14- Public Goods

3 people are deciding how much to contribute to a public park. Each person's contribution is g_i. Each person's utility given the contributions is:

$$a_i ln(g) - g_i$$

We will work two problems:

$a_1 = 100, a_2 = 100, a_3 = 100$

We will work two problems:

$a_1 = 100, a_2 = 200, a_3 = 400$

4.1 Best Response

What is the g_i that maximizes person i's utility given the contributions of the others: g_{-i}? That is, what is their best response function.

Person i's utility is:

$$a_i ln\left(g_i + g_{-i}\right) - g_i$$

Maximize this by finding where the derivative with respect to g_i is zero:

$$\frac{\partial\left(a_i ln\left(g_i + g_{-i}\right) - g_i\right)}{\partial g_i} = 0$$

$$a_i \frac{1}{g_i + g_{-i}} - 1 = 0$$

Best response for person i:

$$g_i = a_i - g_{-i}$$

If $a_1 = a_2 = a_3 = 100$

$$g_i = 100 - g_{-i}$$

What is person i's individually ideal total contribution?

$$100$$

If $a_1 = 100, a_2 = 200, a_3 = 400$

$$g_1 = 100 - g_{-1}$$

$$g_2 = 200 - g_{-2}$$

$$g_3 = 400 - g_{-3}$$

What is person i's individually ideal total contribution?

$$100, 200, 400$$

4.2 Equilibrium

4.2.1 Homogeneous Environment

$a_1 = a_2 = a_3 = 100$

Any set of total contributions that sum to 100 are an equilibrium.

$$(25, 50, 25)$$

$$(100, 0, 0)$$

$$(50, 50, 0)$$

All of these are equilibrium.

4.2.2 Heterogeneous Environment

The only equilibrium is one where the person who cares most (higest a_i) is only one that contributes and they contribute their individually ideal total contributions.

If $a_1 = 100, a_2 = 200, a_3 = 400$

$$(0, 0, 400)$$

4.3 Utilitarian Optimal Total Contributions

What is the g (total contributions) that maximize utilitarian welfare:

We need to write the utilitarian welfare just in terms of g:

$$\frac{a_1 ln\left(g\right) - g_1 + a_2 ln\left(g\right) - g_2 + a_3 ln\left(g\right) - g_3}{3}$$

$$\frac{a_1 ln\left(g\right) + a_2 ln\left(g\right) + a_3 ln\left(g\right) - g_1 - g_2 - g_3}{3}$$

$$\frac{a_1 ln\left(g\right) + a_2 ln\left(g\right) + a_3 ln\left(g\right) - \left(g_1 + g_2 + g_3\right)}{3}$$

$$\frac{\left(a_1 + a_2 + a_3\right) ln\left(g\right) - g}{3}$$

4.3.1 Homogeneous Preferences

$$\frac{300ln\left(g\right)-g}{3}=100ln\left(g\right)-\frac{1}{3}g$$

Maximize this with respect to g:

$$\frac{100}{g}-\frac{1}{3}=0$$

$$\frac{100}{g}=\frac{1}{3}$$

$$300=g$$

4.3.2 Heterogeneous Environment

$$\frac{\left(a_1+a_2+a_3\right)ln\left(g\right)-g}{3}$$

$$\frac{\left(100+200+400\right)ln\left(g\right)-g}{3}$$

$$\frac{700}{3}ln\left(g\right)-\frac{1}{3}g$$

Maximize this:

$$\frac{\partial\left(\frac{700}{3}ln\left(g\right)-\frac{1}{3}g\right)}{\partial g}=0$$

$$\frac{700}{3}\frac{1}{g}-\frac{1}{3}=0$$

$$\frac{700}{3}\frac{1}{g}=\frac{1}{3}$$

$$700\frac{1}{g}=1$$

$$700=g$$

4.4 Utilitarian Ideal Tax

4.4.1 Homogeneous Environment

$$\frac{300}{3} = 100$$

4.4.2 Heterogeneous Environment

$$\frac{700}{3} = 233.333$$

4.5 Individually Idea Taxes

What tax t maximizes person i's utility. That is, what is person i's individually ideal tax?

If we charge everyone t, person i's utility is:

$$a_i \ln(3 * t) - t$$

Maximize this with respect to t:

$$\frac{\partial (a_i \ln(3t) - t)}{\partial t} = 0$$

$$a_i \frac{1}{3t} 3 - 1 = 0$$

$$a_i = t$$

If $a_i = 100$ for everyone, then $t = 100$ for everyone.

For $a_1 = 100, a_2 = 200, a_3 = 400$ the indiviudally ideal taxes are respectively $100, 200, 400$.

4.6 Median Voter's Favorite Tax

$$200$$

5 Chapter 15- Public Goods Mechanisms

Three roommates are deciding whether to buy a tv that costs $c = 1000$. Their valuations are $v_1 = 700, v_2 = 400, v_3 = 200$

For unanimity mechanism, median mechanisms, and VCG mechanism. What is the decision and how much does each person pay?

Is it efficient for them to buy the TV?

Buying the TV is efficient if $\sum_{i=1}^{n} v_i \geq c$. Since it is efficient here, but they don't buy the tv, the Unanimity mechanism chooses the inefficient outcome.

5.1 Unanimity

If everyone's valuation is $v_i \geq \frac{c}{n}$ then the public good is provided and each person pays $\frac{c}{n}$.

$v_1 = 700, v_2 = 400, v_3 = 200$

Since $v_3 < \frac{1000}{3} = 333.333$ they don't buy the TV no one pays anything.

5.1.1 Another Example

$v_1 = 700, v_2 = 400, v_3 = 400$

$c = 1000$

They buy the TV and everyone pays 333.33..

5.2 Median Mechanism

$c = 1000$

$v_1 = 700, v_2 = 400, v_3 = 200$

If the median voter's valuation $v_i \geq \frac{c}{n}$ the public good is provided and everyone pays $\frac{c}{n}$.

$v_2 = 400 \geq \frac{1000}{3}$ they buy the TV and everyone pays $\frac{1000}{3}$.

5.2.1 Another Example

$c = 1000$

$v_1 = 700, v_2 = 300, v_3 = 200$

Since $v_2 < \frac{1000}{3}$ they don't buy the TV.

5.3 VCG

The VCG mechanism **always** makes the efficient decision, but uses a rather complex way of calculating amount paid.

Payments are only paid by **pivotal** people and they amount they pay is the **net cost** they impost on society.

Someone is **pivotal** if their existence changes whether it is efficient to produce/buy the public good.

$c = 1000$

$v_1 = 700, v_2 = 400, v_3 = 200$

The VCG Mechanism chooses for them to buy the TV.

If the public good is provided, only pivotal people pay.

Who is pivotal in this model?

Person 1 is pivotal since if they did not exist, then $v_2 + v_3 = 600 < 1000$ and they wouldn't buy the TV.

Person 2 is pivotal since if they did not exist, then $v_1 + v_3 = 700 + 200 = 900 < 1000$ and they wouldn't buy the TV.

Person 3 is not pivotal since even they did not exist, it would still be efficient to buy the TV. $700 + 400 = 1100 > 1000$.

Amount 1 and 2 pay are the **net costs they impose on society.**

Let $v_{-i} = \left(\sum_{j=1}^{n} v_j\right) - v_i$

$$t_i = c - v_{-i}$$

$c = 1000$

$v_1 = 700, v_2 = 400, v_3 = 200$

$$t_1 = 1000 - (v_2 + v_3) = 1000 - 600 = 400$$

$$t_2 = 1000 - (700 + 200) = 1000 - 900 = 100$$

$$t_3 = 0$$

Is there a budget deficit?

Yes, they collect only 500 but the TV costs 1000.