8/29/2024

1 Optimization Exercises

1.1 A.1

Maximize the function $f(x) = -x^2 + 4x + 4$.
Find where this function has zero slope:

$$\frac{\partial\left(-x^2 + 4x + 4\right)}{\partial x} = 4 - 2x$$

$$4 - 2x = 0$$

$$4 = 2x$$

$$x = 2$$

1.2 A.2

Maximize the function $f(x) = ln(x) - \frac{1}{4}x + 4$.
Where is the slope of this equal to zero?

$$\frac{\partial\left(\log(x) - \frac{1}{4}x + 4\right)}{\partial x} = \frac{1}{x} - \frac{1}{4}$$

$$\frac{1}{x} - \frac{1}{4} = 0$$

$$\frac{1}{x} = \frac{1}{4}$$

$$x = 4$$

1.3 A.3

Maximize the function $f(x, y) = -x^2 - y^2 + 2x + 2y$.

We need to find an x and y where the slope of f is zero in both directions at the same time.

$$\frac{\partial\left(-x^2 - y^2 + 2x + 2y\right)}{\partial x} = 0$$

$$\frac{\partial\left(-x^2 - y^2 + 2x + 2y\right)}{\partial y} = 0$$

Taking these derivatives:

$$2 - 2x = 0$$

$$2 - 2y = 0$$

Solve these:

$$x = 1$$

$$y = 1$$

1.4 A.4

Maximize the function $f(x, y) = x + y$ subject to the constraint $x + 2y \leq 60$.

With the Lagrange function, we set up a penalized version of the maximization problem.

Let's approach the problem intuitively.

If you have a linear problem, the optimal solution will always be an extreme (where one of the variables is zero).

$$x = 60$$

$$L(x, y) = x + y - \lambda(x + 2y - 60)$$

$$\frac{\partial\left(x + y - \lambda(x + 2y - 60)\right)}{\partial x} = 0$$

$$\frac{\partial\left(x + y - \lambda(x + 2y - 60)\right)}{\partial y} = 0$$

$$\frac{\partial\left(x+y-\lambda\left(x+2y-60\right)\right)}{\partial\lambda}=0$$

Try to solve these:

$$1-\lambda=0$$

$$1-2\lambda=0$$

$$-x-2y+60=0$$

$$\lambda=1$$

$$\lambda=\frac{1}{2}$$

1.5 A.5

Maximize $f(x,y)=xy$ subject to the constraint $x+2y\leq 60$.

$$x+2y-60\leq 0$$

$$L\left(x,y\right)=xy-\lambda\left(x+2y-60\right)$$

Maximize this function unconstrained with respect to x,y,λ

$$\frac{\partial\left(xy-\lambda\left(x+2y-60\right)\right)}{\partial x}=0$$

$$\frac{\partial\left(xy-\lambda\left(x+2y-60\right)\right)}{\partial y}=0$$

$$\frac{\partial\left(xy-\lambda\left(x+2y-60\right)\right)}{\partial\lambda}=0$$

Take all derivatives:

$$y-\lambda=0$$

$$x - 2\lambda = 0$$

$$60 = x + 2y$$

Solve this system of equations:

$$y = \lambda$$

$$x - 2\left(y\right) = 0$$

$$x = 2y$$

Then plug this relationship into the third one from above (the constraint)

$$60 = \left(2y\right) + 2y$$

$$4y = 60$$

$$y = 15$$

$$x = 2\left(15\right) = 30$$

1.6 A.6

Maximize the function $f(x, y) = x^{\frac{1}{2}} + y^{\frac{1}{2}}$ subject to the constraint $x + 2y \leq 60$. **Try this one at home.**

$$L\left(x, y\right) = x^{\frac{1}{2}} + y^{\frac{1}{2}} - \lambda\left(x + 2y - 60\right)$$

$$\frac{\partial\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} - \lambda\left(x + 2y - 60\right)\right)}{\partial x} = \frac{1}{2\sqrt{x}} - \lambda$$

$$\frac{\partial\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} - \lambda\left(x + 2y - 60\right)\right)}{\partial y} = \frac{1}{2\sqrt{y}} - 2\lambda$$

$$\frac{\partial\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}-\lambda\left(x+2y-60\right)\right)}{\partial\lambda} = -x-2y+60$$

$$\frac{1}{2\sqrt{x}}=\lambda$$

$$\frac{1}{2\sqrt{y}}-2\left(\frac{1}{2\sqrt{x}}\right)=0$$

$$\frac{1}{2\sqrt{y}}=\frac{1}{\sqrt{x}}$$

$$\sqrt{x}=2\sqrt{y}$$

$$x=4y$$

$$60=4y+2y$$

$$y=10$$

$$x=40$$

1.7 A.7

Maximize the function $f(x,y)=min\{x,y\}$ subject to the constraint $x+2y\leq 60$. Write down the "no waste condition"

$$x=y$$

...and the constraint

$$x+2y=60$$

Solve these:

$$x+2\left(x\right)=60$$

$$3x=60$$

$$x=20$$

$$y=20$$

1.8 A.8

Maximize $100 - (x-10)^2 - (y-10)^2$ subject to the constraint $x + 2y \leq 10$
Try this one at home.