0.1 Exercises about Majority Rule and Copeland's Method.

$1 : a \succ b \succ c$

$2 : a \succ b \succ c$

$3 : b \succ c \succ a$

$4 : b \succ c \succ a$

$5 : c \succ a \succ b$

Majority Rule.

We compare every pair of outcomes. And if at least 50% of people prefer x to y then $x \succsim^* y$.

Compare a and b: 3 votes for a and 2 votes for b. a wins $a \succ^* b$.

Compare a and c: c wins (with 3 of 5 votes) $c \succ^* a$

Compare b and c: b wins (with 4 of 5 votes) $b \succ^* c$.

We have a cycle in the preferences. $a \succ b, b \succ c, c \succ a$ and thus, the social preferences are intransitive.

Copeland's Method

Compare a and b: 3 votes for a and 2 votes for b. a wins and get 1 point.

Compare a and c: c wins (with 3 of 5 votes) and gets 1 point.

Compare b and c: b wins (with 4 of 5 votes) and gets 1 point.

$a : 1, b : 1, c : 1.$

$$a \sim^* b \sim^* c$$

1 Properties of Preference Aggregation Rules

1.1 Basic Properties

What should a preference aggregation rule achieve?

1. *Complete*. The preference aggregation rule is **complete**, if the social preferences are always complete for any set of individual preferences.

2. Transitive. The preference aggregation rule is **transitive**, if the social preferences are always transitive for any set of individual preferences.

3. Pareto Efficient. If everyone strictly prefers x to y then so does the social preference. If for everyone $x \succ_i y$ then $x \succ^* y$.

1.2 What we know so far.

Rule	Complete	Transitive	Pareto
Dictatorship	✓	✓	✓
Unanimity Rule	×	✓	✓
Majority Rule	✓	×	✓

Is Majority rule pareto efficient?

The definition. If everyone strictly prefers x to y then so does the social preference.

For majority rule if $> 50\%$ of people prefer x to y then $x \succ^* y$.

If everyone prefers x to y then 100% of people will vote for x and so $x \succ^* y$.

1.3 Methods that use a Score

Any method that assigns a score to the outcomes and then ranks the outcomes by score will always be complete and transitive.

1.4 Copeland's Method

Complete- yes, because it assigns scores.

Transitive- yes, because it assigns scores.

Pareto Efficient-

Suppose everyone prefers x to y. Does Copeland's method $x \succ^* y$? That is does x get a strictly higher score than y?

Anyone who likes y better than z better also like x better than z.

$$y \succ_i z$$

Since everyone prefers x to y we have. Anyone who prefers y to z also prefers x to z since **everyone** prefers x to y.

$$x \succ_i y \succ_i z$$

If a majority of people prefer y to z then a majority of people will also prefer x to z.

Any pairwise competition that y wins, x will also win. Plus x beats y. Thus, the score of x is **at least** one more than the score of y. Thus $x \succ^* y$.

Rule	Complete	Transitive	Pareto
Dictatorship	✓	✓	✓
Unanimity Rule	×	✓	✓
Majority Rule	✓	×	✓
Copelands	✓	✓	✓

1.5 Example of Pareto Efficiency in Copeland's Rule

$1 : a \succ b \succ c$

$2 : a \succ b \succ c$

$3 : b \succ c \succ a$

$4 : b \succ c \succ a$

$5 : b \succ a \succ c$

b beats a- b gets a point

a beats c- a gets a point

b beats c- b gets a point

$b \succ^* a \succ^* c$

1.6 Borda Method

Complete- yes, because it assigns scores.

Transitive- yes, because it assigns scores.

Pareto Efficient-

If everyone strictly prefers x to y. Then x gets a strictly higher score for each person than y does. So of course the sum of the scores for x has to be strictly higher than y and so $x \succ^* y$.

Rule	Complete	Transitive	Pareto
Dictatorship	✓	✓	✓
Unanimity Rule	×	✓	✓
Majority Rule	✓	×	✓
Copelands	✓	✓	✓
Borda Count	✓	✓	✓

1.7 Independence of Irrelevant Alternatives

$1 : a \succ b \succ c$

$2 : b \succ c \succ a$

$3 : c \succ a \succ b$

Borda:

$a : 3 + 2 + 1$

$b : 2 + 3 + 1$

$c : 1 + 2 + 3$

$a \sim^* b \sim^* c$

Let's focus on $a \sim^* b$

Swap a and c for Person 2.

$1 : a \succ b \succ c$

$2 : b \succ a \succ c$

$3 : c \succ a \succ b$

$a : 3 + 2 + 2 = 7$

$b : 2 + 3 + 1 = 6$

$c : 1 + 1 + 3 = 4$

$a \succ^* b \succ^* c$

Even though everyone who likes $a \succ b$ in example 1 still does and everyone who likes $b \succ a$ in example 1 still does, the social preference over a and b changed from $a \sim^* b$ to $a \succ^* b$.

1.8 IIA

A preference aggregation rule obeys **Independence of Irrelevant Alternatives** [IIA] if for any two sets of preferences where the preference for a and b is the same between the two sets, they should have the same social preference between a and b.

1.9 Why Does this Matter?

1.9.1 Borda Example

25 People: $a \succ b \succ c$

40 People: $b \succ c \succ a$

35 People: c$\succ a \succ b$

Borda:

$a : (25)\,3 + (40)\,1 + (35)\,2 = 185$

$b : (25)\,2 + (40)\,3 + (35)\,1 = 205$

$c : (25)\,1 + (40)\,2 + (35)\,3 = 210$

$c \succ^* b \succ^* a.$

25 People: $b \succ c$

40 People: $b \succ c$

35 People: c$\succ b$

$b : (25)\,2 + (40)\,2 + (35)\,1 = 165$

$c : (25)\,1 + (40)\,1 + (35)\,2 = 135$

$b \succ^* c$

1.10 Arrow's

Rule	Complete	Transitive	Pareto	IIA
Dictatorship	✓	✓	✓	✓
Unanimity Rule	×	✓	✓	✓
Majority Rule	✓	×	✓	✓
Copelands	✓	✓	✓	×
Borda Count	✓	✓	✓	×

Statement: If there are at least three options available, the **only** preference aggregation rule that is complete, transitive, Pareto efficient, and respects IIA is a **dictatorship.**