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1 Binary Relations

This is the most general way to define "relationships" formally.

- "Is a sibling on" on the set of people.
- "Is at least big as" on the set of numbers.
- "Is taller than" on the set of people.
- "Shares a border with" on the set of countries.

A binary relation is way of defining **relationship** among things in a **set**.

Formally a relation R on the set X is:

$$R \subseteq X \times X$$

$X \times X$ is the Cartesian product of X with itself. This is the set of all ordered pairs of elements of X.

For example:

$X = \{a, b, c\}$.

$$X \times X = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

R says "of all of the possible pairs of things than could have this relationship, which actually do?"

Suppose R is "comes strictly earlier in the alphabet" on the set $X = \{a, b, c\}$.

$$R = \{(a, b), (a, c), (b, c)\}$$

Suppose R is "comes at least as early in the alphabet".

$$R = \{(a, b), (a, c), (b, c), (a, a), (b, b), (c, c)\}$$

Suppose $\geq$ is the "at least as large" relation on $\{1, 2, 3\}$.

$$= \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 1), (3, 2)\}$$

When some pair, like $(2, 1)$ is part of the relation, we can also write:

$$2 \geq 1$$

We can do this for other relations as well. Like in the "comes strictly earlier in the alphabet" relation we might write:

$$aRb$$

Example. Strictly taller T on the set $\{greg, christina, michael\}$, $X = \{g, c, m\}$

$$gTc, mTg, mTc$$

Example. At least as tall T on the set $\{greg, christina, michael\}$, $X = \{g, c, m\}$

$$gTc, mTg, mTc, gTg, cTc, mTm$$

1.1 Preferences

Suppose we want to represent "preferences". We will use the $\succsim$ to represent "weak preference".

$x \succsim y$ means "x is at least as good a y".

Suppose apples are best, the bananas, then carrots $X = \{a, b, c\}$ this is the preference relation:

$$a \succsim b, b \succsim c, a \succsim c, a \succsim a, b \succsim b, c \succsim c$$

Let's define a preference relation over bowls of ice cream. X is the set of bowls of ice cream where v is the amount of vanilla and c is the amount of chocolate. (v, c) is a bowl of ice cream. $(1, 1)$ is a scoop of each. $(1, 0)$ is a scoop of vanilla, $(0, 2)$ is two scoops of chocolate, $(8, \pi)$ with 8 scoops of vanilla and 3.141.... scoops of chocolate.

Suppose someone prefers more ice cream to less. And, if two bowls have the same amount, they prefer the one with more chocolate. Then for example:

$$(2, 0) \succsim (1, 0), (1, 1) \succsim (1, 0), (0, 2) \succsim (2, 0)$$

But, for example:

$$(2, 0) \not\succsim (0, 2)$$

1.2 Properties of Relations

1.2.1 Reflexive

Everything in X is related to itself.

We say a relation $\succsim$ is **reflexive** if:

$$\forall x \in X, x \succsim x$$

Examples.

$>$ *Not Reflexive* (Counter-example: $5 \not> 5$)

$\geq$ *Reflexive*

$=$ *Reflexive*

Same Parents As. *Reflexive*

At least as Tall As. *Reflexive*

Strictly Taller Than. *Not Reflexive.*

Complete A relation that has something to say above every pair. For every possible pair, at least one direction must be true (even for pair of the same thing twice).

Formally, we say the relation $\succsim$ is **complete** if:

$$\forall x, y \in X, x \succsim y \ or \ y \succsim x \ (or\, both)$$

Examples

$>$ *Not Complete* (Counter-Example $5 \not> 5$)

$\geq$ *Complete*

$=$ *Not Complete* (Counter-Example $5 \neq 7$)

$\leq$ *Complete*

Sibling *Not Complete*

Same Parents As *Not Complete*

At least as Tall As *Complete*

Strictly Taller Than *Not Complete*

Any relation that is not reflexive is not complete.

1.3 Transitivity

Preferences are transitive if when $a \succsim b$ and $b \succsim c$ then $a \succsim c$.

Formally:

$$\forall x, y, z \in X, x \succsim y \& y \succsim z \Rightarrow x \succsim z$$

$>$ *Transitive*

$\geq$ *Transitive*

$=$ *Transitive*

Sibling (*True of Biological but not Step Siblings*)

Same Parents As *Transitive*

At least as Tall *Transitive*

Strictly Taller Than *Transitive*

Example of a non-transitive relation:

B is "Beats in a game of rock paper scissors." on the set $\{r, p, s\}$.

$$rBs, sBp, pBr$$

1.4 Why transitivity matters.

Let's suppose these are preferences

$$a \succsim b, b \succsim c, c \succsim a$$

You would never be able to choose something from the menu $\{a, b, c\}$... there is nothing in this set that is at least as good as every other thing! a is not as good as c, b is not as good as a, c is not as good as b. What should you choose?

1.5 Rational Preferences

A we a person is rational if they have a preference relation and it is **complete, and transitive.**