Microeconomics Lecture Notes

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Contents

Part I

Choice

1 Objects and Sets

In this chapter, we begin our study of how to model the choices of individuals. The first step is to define what "things" are relevant in the model, and what things are actually available to a consumer.

1.1 Choice Sets

In a model of choice, we need to be clear about what the consumer is choosing. We do this by defining the **set** of objects that the consumer might choose among in the model. A common name for this is the **choice set,** but these notes will also use **Universe of Choice Objects**. In these notes, the elements of X **choice objects**.

Choice sets / Universe of Choice Objects: X
Bundles / Choice Objects: $x : x \in X$

Example 1. Ice Cream Bowls. $X = \{$*a scoop of vanilla, a scoop of chocolate, two scoops of vanilla, two scoops of chocolate, a scoop of each flavor*$\}$.

We could equally well denote these bowls of ice cream with vectors indicating the number of scoops of each flavor. When we write choice objects as vectors of underlying types of things, we call them **bundles**. We refer to each type of thing as a **good**. In the previous example we have two goods: vanilla ice cream and chocolate ice cream.

Example 2. Ice Cream Bowls as Bundles. $X = \{(1,0), (0,1), (2,0), (0,2), (1,1)\}$.

There is no reason to limit the scope of our models here; we could probably expand the choice set to $X = \mathbb{R}^2_+$. That is, all the two dimensional vectors with non-negative amounts of each good.

Example 3. All Real Ice Cream Bowls. $X = \mathbb{R}^2_+$.

1.2 Assumptions about X.

When X is a finite set, there's nothing more we need to do to ensure suitable models can be built from it. When X is infinite, we can make some assumptions that will help ensure that our models don't fall apart. Unless otherwise mentioned, for the rest of these notes, we will make the following assumptions.

Euclidean: $X \subseteq \mathbb{R}^n_+$.
Non-empty: $X \neq \emptyset$.
Closed: X is closed.
Convex: X is convex.

The last two assumptions above are worth remark. Briefly, if X is not closed, then consumers might find no best choice the same way there is no biggest number in the set $[0, 1)$. If X is not convex we can't use arguments like "between these two bundles there must be some other bundle such that...".

1.3 Budget Set B

The particular bundles available to some consumer are collectively called the **budget set** and denoted B.

Budget: $B \subseteq X$**.**

A common type of budget set is the **competitive budget set**. This is a budget set induced on a set of bundles over some number of goods by having fixed prices per unit of those goods and a budget. For a two-good case, we can define it formally this way:

Competitive Budget for n Goods: $B(p, m) = \{x \mid x \in X, \sum_{i=1}^{n} p_i x_i \leq m\}$ **.**

2 Preferences

2.1 The Preference Relation

Now that we have defined what exists in the model, we are ready to define the objective by which the consumer makes choices. This is done using the **preference relation** $\succsim$. $\succsim$ is a binary relation. Formally, a binary relation on a set X is a subset of the Cartesian product X with itself. That is, it always compares pairs of bundles.

A Relation on X: $\succsim \subseteq X \times X$

We have two ways of representing what pairs of bundles are in the relation $\succsim$.

Set Notation: $(x, y) \in \succsim$**.** **Infix Notation:** $x \succsim y$**.**

When $\succsim$ is a preference relation, $(x, y) \in \succsim$ or $x \succsim y$ both mean "bundle x is at least as good as y" or "bundle x is weakly preferred to y".

2.2 Properties of Relations

Relations can have many properties. Here are some that are useful to know about. *These are not important to memorize.* It is more important that you understand how to work with these kinds of definitions.

Reflexive: $\forall x \in X, x \succsim x$.
Not Reflexive: $\exists x \in X, \neg(x \succsim x)$.
Irreflexive: $\forall x \in X, \neg(x \succsim x)$.
Symmetric: $\forall x, y \in X, x \succsim y \Longrightarrow y \succsim x$.
Not Symmetric: $\exists x, y \in X, x \succsim y \,\&\, \neg(y \succsim x)$.
Antisymmetric: $\forall x, y \in X \; s.t. \; x \neq y, x \succsim y \implies \neg(y \succsim x)$.
Asymmetric: $\forall x, y \in X, x \succsim y \implies \neg(y \succsim x)$.
Complete: $\forall x, y \in X, x \succsim y \vee x \succsim y$.
Incomplete: $\exists x, y \in X, \neg(x \succsim y) \wedge \neg(y \succsim x)$.
Transitive: $\forall x, y, z \in X, x \succsim y \,\&\, y \succsim z \implies x \succsim z$.
Negatively Transitive: $\forall x, y, z \in X, \neg(x \succsim y) \,\&\, \neg(y \succsim z) \implies \neg(x \succsim z)$.

2.3 Induced Relations.

Exercise 4. Every relation $\succsim$ can be decomposed into a symmetric $\sim$ and asymmetric relation $\succ$ such that $\sim \cup \succ = \succsim$.

Indifference Relation: $\sim = \{(x, y) \,|\, (x, y) \in X \times X, (x, y) \,\&\, (y, x) \in \succsim\}$.
Strict Preference Relation: $\succ = \{(x, y) \,|\, (x, y) \in X \times X, (x, y) \in \succsim, (y, x) \notin \succsim\}$.

With these definitions in place, we can define few new properties more easily[1] that relate to the property called **transitivity.**

Quasitransitive: $\forall x, y, z \in X, x \succ y \,\&\, y \succ z \Rightarrow x \succ z$.
Acyclic: $\forall x_1, x_2, ..., x_n \in X$, $x_1 \succ x_2 \,\&\, x_2 \succ x_3 \,\&\, ... \,\&x_{n-1} \succ x_n \Rightarrow x_1 \succsim x_n$.

2.4 Logical Relationships

There are many logical relationships between the various relation properties. Some of these relationships are straight-forward, some are more complex.

2.5 Rational Relations

Most of economics relies on two assumptions about $\succsim$. When $\succsim$ meets these assumptions, we call is a **rational preference relation**.[2]

Complete: $\forall x, y \in X, (x \succsim y) \vee (y \succsim x)$.
Transitive: $\forall x, y, z \in X, x \succsim y \wedge y \succsim z \implies x \succsim z$.

[1] We can define these properties without using $\succ$ or $\sim$, but the definitions are more verbose. For instance, quasitransitivity can be defined using only $\succsim$ simply by expanding the definition of $\succ$ like this: $\forall x, y, z \in X, x \succsim \,\&\, \neg(y \succsim x) \,\&y \succsim z\&\neg(z \succsim x) \implies x \succsim z\&\neg(z \succsim x)$.

[2] Some texts also include: **Reflexive**: $\forall x \in X, x \succsim x$, but it is implied by completeness.

We have names for relations that meet certain combinations of properties. Here are some of the names that apply to orders that are both reflexive and transitive:

Preorder: $\succsim$ is reflexive, transitive.
Possibly Incomplete:
Equivalence Relation: $\succsim$ is reflexive, **symmetric**, transitive.
Partial Order: $\succsim$ is reflexive, **anti-symmetric**, transitive.
Complete Orders:
Weak Order: $\succsim$ is reflexive, transitive, **complete**.
Linear Order: $\succsim$ is reflexive, anti-symmetric, transitive, **complete**.

Of these, When a relation is complete and transitive, we say that $\succeq$ is a "weak order". **Why do we make these assumptions?** Let's see.

2.6 From Preferences to Choice

Preferences are the objective used to make choices. We now define choice formally using a **choice function**. The choice function is a mapping from the set of non-empty budget sets to the powerset of X. That is, for any budget, the choice correspondence gives the set of acceptable choices from X.

Choice Function: $C : P(X) \to P(X)$ such that $C(B) \subseteq B$ for all $B \subseteq X$

We can define a choice function explicitly or induce one from preferences this way by assuming a consumer is willing to choose any $x \in B$ that is at least as good as everything else in B.

Choice Function Induced by $\succsim$: $C(B) = \{x \mid x \in B, (\forall x' \in B : x \succeq x')\}$.

Example 5. Choice Function Induced by Rational Preferences. Suppose $X = \{a, b, c\}$, and $\succsim = \{(a, b), (a, c), (b, c), (c, b), (a, a), (b, b), (c, c)\}$. $C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c, C(\{a, b\}) = a, C(\{a, c\}) = a, C(\{b, c\}) = \{b, c\}, C(\{a, b, c\}) = a$.

Notice here that every budget has a non-empty choice set. This is not always the case.

Example 6. Choice Function Induced by Non-Transitive Preferences. Suppose $X = \{a, b, c\}$, and $\succsim = \{(a, b), (b, c), (c, a), (a, a), (b, b), (c, c)\}$. $C(\{a, b, c\}) = \emptyset$.

In the above example, the consumer cannot make a choice from the set $\{a, b, c\}$. This is because $a \succ b, a \succ c, c \succ a$. We have a strict cycle in preference.

2.7 Cycles Lead to Empty Choice Sets

Proposition 7. For finite X, Empty Choice Is Equivalent to a Cycle in $\succ$. *For complete and reflexive preference relations, the existence of a set of distinct bundles $\{x_1, ..., x_n\}$ such that $x_1 \succ x_2 \succ ... \succ x_n \succ x_1$ is equivalent to the existence of a finite budget B such that $C(B) = \emptyset$.*

Proof. To prove the $\Rightarrow$ direction, take the set $\{x_1, ..., x_n\}$ to be the budget. $C(\{x_1, ..., x_n\}) = \emptyset$, since the strict preference cycle ensures for every $x \in B$ there is some $y \in B$ such that $x \not\succsim y$.

The proof of $\Leftarrow$ is as follows. Since preferences are reflexive and complete, any budget set for which $C(B) = \emptyset$ has to have at least three elements. Suppose one exists. Take any element in B and call it x_1. There must be something that x_1 is not as good as, call it x_2. By completeness, $x_2 \succ x_1$. The same goes for x_2. We can find an x_3 such that $x_3 \succ x_2 \succ x_1$. Continue constructing this chain until its length is greater than the number of elements of B. The chain is now $x_n \succ ... \succ x_1$. Since there are more elements in the chain than elements in B, some element must appear twice. Let x_i be the first such element appearing twice. The subsequence $x_i \succ ... \succ x_i$ is a cycle in $\succ$. $\square$

Proposition 8. Rational $\succsim$ Rules Out Empty Choice. *If $\succsim$ is Rational, $\nexists B \in P(X)/\emptyset : C(B) \neq \emptyset$.*

Proof. Since $\succsim$ is rational, $\succ$ is transitive. Since $\succ$ is asymmetric by construction, this implies it has no cycles. Thus, by the contrapositive Proposition 7 there does not exist a $B \in P(X)/\emptyset$ such that $C(B) = \emptyset$. $\square$

We can rule out empty choice by ensuring there is no cycle in $\succ$. This is implied by transitivity of $\succsim$, but does not imply transitivity of $\succsim$. Here is a counter-example: $x \succ y, y \sim z, z \succ x$. So transitivity of $\succsim$ appears to be stronger than we need. Why do we assume it instead of just assuming there are no cycles in $\succ$?

2.8 Rational Preferences and Non-Empty, Coherent Choice

In the previous section we looked at non-empty choice in a setting where X is finite. When X is infinite, it would be too much to ask for *every* $B \subseteq X$ to have a non-empty choice set. This is because B may well be open. But we can restrict our attention to finite B and ask what rational $\succsim$ buys us in terms of choice. As hinted at by the previous section, rational $\succsim$ is stronger than we need for *just* non-emptiness of choice. In fact, it gets us something else: **coherent choice**.

Finite Non-empty C: *For any finite B where $B \neq \emptyset$, $C(B) \neq \emptyset$* **Coherent** C: *For every x, y and B, B' such that $x, y \in B \cap B'$, $x \in C(B) \wedge y \notin C(B) \Rightarrow y \notin C(B')$.*

Proposition 9. Rational $\succsim$ implies Finite-nonempty and Coherent Choice. *If $\succsim$ is rational, then C is coherent and finite-nonempty.*

To understand coherent choice, let's look at an example violating it.

Example 10. Minimal Incoherent Choice Example. $X = \{x, y, z\}$ and $C(\{x, y\}) = x, C(\{x, y, z\}) = \{y\}$.

Notice x is chosen over y in the first bundle. However, the addition of z makes the consumer choose y instead. That is pretty weird.

2.9 Contour Sets

The relation $\succsim$ is about ordered pairs from X, but we can use the relation to create some useful sets in X. These are called **contour sets**.

Upper Contour Set: $\succeq(x) = \{y | y \in X \,\& y \succsim x\}$.
Lower Contour Set: $\precsim(x) = \{y | y \in X \,\& x \succsim y\}$.
Indifference Set: $\sim(x) = \{y | y \in X \,\& x \sim y\}$.

The upper contour set and lower contour sets are sometimes called the "no better than set" and the "no worse than set". We can also define $\succ(x)$ and $\prec(x)$ analogously as the interior of $\succsim(x)$ and $\precsim(x)$ respectively.

2.10 Indifference Curves

We use indifference sets extensively to visualize and study the structure of preferences. This is because in most of the cases we'll look at, $\sim(x)$ is one dimension smaller than X. That is, when X is two-dimensional, $\sim(x)$ is a one-dimensional *manifold.* Because of this, we often call $\sim(x)$ an **indifference curve**.

Indifference curves need not have any special structure, unless we make further assumptions about preferences. There is only one important result to know:

Proposition 11. Indifference Curves do not Cross. *If $\succsim$ is rational, $\forall x, y \in X : x \neq y \wedge x \succ y, \sim(x) \cap \sim(y) = \emptyset$.*

Proof. Suppose otherwise, then there exists a bundle $z \in X$ such that $z \in \sim(x)$ and $z \in \sim(y)$. Thus, $y \sim z$. By assumption, $x \succ y$. Thus, $x \succ y \sim z$. By the result proved in *Exercise 2.9*, $x \succ z$ which contradicts that $z \in \sim(x)$ $\square$

2.11 Exercises

Prove each of the following. For 2.1-2.3, assume the sets are relations as defined in Section 2.9.

2.1. $\succ \cup \sim = \succeq \;\&\; \succ \cap \sim = \emptyset$

2.2. $\sim(x) = \succeq(x) \cap \preceq(x)$.

2.3. $\succeq(x) = \sim(x) \cup \succ(x)$.

2.4. Every relation $\succsim$ can be decomposed into a symmetric $\sim$ and asymmetric relation $\succ$ such that $\sim \cup \succ = \succsim$.

2.5. A transitive $\succsim$ is quasi-transitive.

2.6. A quasi-transitive $\succsim$ is acyclic.

2.7. Rational $\succsim$ implies transitive $\sim$.
2.8. Rational $\succsim$ implies transitive $\succ$.
2.9. Rational $\succsim$ implies that if $x \succ y \sim z$ then $x \succ z$.

2.10. If $\succsim$ is rational then $\forall x, x' \in X$*:* $\succeq(x) \subseteq \succeq(x') \vee \succeq(x') \subseteq \succeq(x)$.

3 Preferences as Directed Graphs

3.1 Directed Graphs

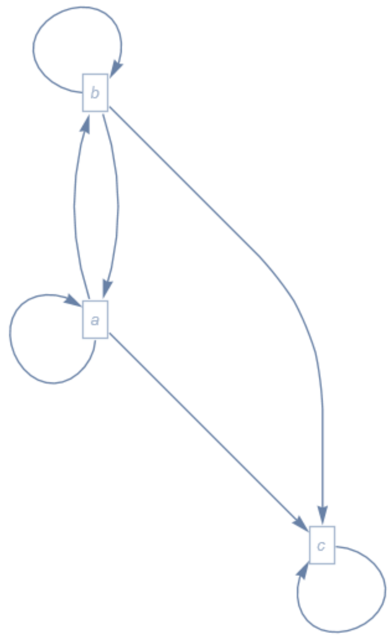

Figure 3.1: Digraph for preferences in Example 5.

A **directed graph** or **digraph** is a pair $G = \{X, E\}$ consisting of **vertices** X along with **directed edges/arcs** $E \subseteq X \times X$**.** Directed graphs can be used to represent a preference relation $\succsim$. By letting $G = \{X, \succsim\}$.

$$\boxed{\textbf{Digraph Induced by } \succsim\text{: } G = \{X, \succsim\}.}$$

An example of a digraph representing the preferences in Example 5 is show in Figure 3.1.

There are several concepts from graph theory that are useful.

Self-loop: A directed edge from a vertex to itself.
Path: A sequence of vertices and edges in G: $P = (x_1, E_1, x_2, E_2, ..., E_{n-1}, x_n)$ such that $E_i = (x_i, x_{i+1})$.
Cycle: A *path* such that $x_1 = x_n$.
Acyclic Graph: A graph with no cycles.
Length of Path: The length of path P is the number of edges in P.
Adjacent: Vertex x is adjacent to y if there is a path of length 1 in G.
Subgraph: $\tilde{G} = \left\{\tilde{X}, \tilde{E}\right\}$ is a subgraph of G if it is a graph and $\tilde{X} \subseteq X$ $\tilde{E} \subseteq E$.
Complete: G is complete if every pair of distinct vertices has two directed edges and every vertex has a self-loop.
In-Degree: The in-degree of a vertex x is the number of edges of the form (x', x) (self-loops count).
Source: A vertex with in-degree zero.
Out-Degree: The out-degree of a vertex x is the number of edges of the form (x, x') (self-loops count).
Sink: A vertex with out-degree zero.

One type of useful subgraphs are **vertex-induced subgraphs**.

Definition 12. Vertex-Induced Subgraph. Let $G = \{X, E\}$. For any $S \subseteq X$, $G[S] = \left\{S, \tilde{E}\right\}$ where $\tilde{E} = \{(x, y) \,|x, y \in S \wedge (x, y) \in E\}$.

Figure 3.2: The vertex-induced subgraph $G[\{a, c\}]$ from the graph G in Figure 3.1.

3.2 Mapping Properties of $\succsim$ into Graph Properties

We have looked at several properties of preference relations. Each of these properties is characterized by something about the digraph it induces.

Reflexive $\succsim$ $\Leftrightarrow$ Every vertex has a self-loop.
Not Reflexive $\succsim$ $\Leftrightarrow$ There is some vertex that does not have a self-loop.
Irreflexive $\succsim$ $\Leftrightarrow$ No vertex in has a self-loop.
Symmetric $\succsim$ $\Leftrightarrow$ There is either zero or two directed edges between every pair of distinct vertices and every vertex has a self-loop.
Not Symmetric $\succsim$ $\Leftrightarrow$ There is some pair of distinct vertices that has only a single directed edge or a vertex that lacks a self-loop.
Asymmetric $\succsim$ $\Leftrightarrow$ There is no more than one directed edge between every pair of distinct vertices and there are no self-loops. A graph with this property is called **Oriented**.
Antisymmetric $\succsim$ $\Leftrightarrow$ There is no more than one directed edge between every pair of distinct vertices.
Complete $\succsim$ $\Leftrightarrow$ There is at least one directed edge between every pair of vertices.
Incomplete $\succsim$ $\Leftrightarrow$ There is some pair of vertices that has no edge.

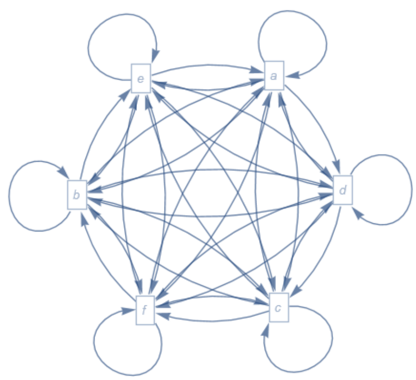

Figure 3.3: A Complete Digraph on $X = \{a, b, c, d, e, f\}$.

Rational preferences have a particular structure to them that can be visualized with a graph. Rational preference induce a graph that is a made up of complete subgraphs $G_1, ..., G_n$ such that for every $i, j \in \{1, ..., n\}$ such that $i < j$ and every $x \in G_i$ and $y \in G_j$ there is a single directed edge (x, y) (from x to y). An example of such a graph on $X = \{a, b, c, d, e\}$ is shown below where $V_1 = \{a, b, c\}$ and $V_2 = \{d, e\}$.

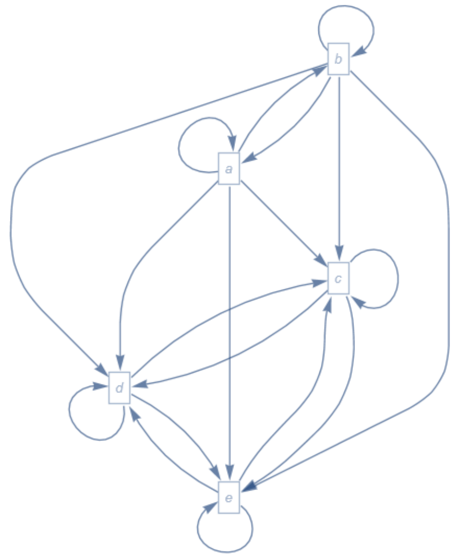

Figure 3.4: Graph induced by a complete and transitive relation on $X = \{a, b, c, d, e\}$.

3.3 Empty-Choice and Cycles as a Graph Theory Proof

In the previous chapter we looked at the following result: *For complete and reflexive preference relations, the existence of a set of distinct bundles* $\{x_1, ..., x_n\}$ *such that* $x_1 \succ x_2 \succ ... \succ x_n \succ x_1$ *is equivalent to the existence of a finite budget* B *such that* $C(B) = \emptyset$.

Let's translate this proposition into graph theory. First, let $G_{\succ} = \{X, \succ\}$. This the graph of the strict induced relation. The first statement (*a set of distinct bundles* $\{x_1, ..., x_n\}$ *such that* $x_1 \succ x_2 \succ ... \succ x_n \succ x_1$) is equivalent to the existence of a cycle in $G_{\succ}$. To translate the second statement, first note that for any complete relation $x \in C(B)$ if and only if $\nexists y \in B$ such that $y \succ x$. Thus, in terms of the graph of $G_{\succ}$, $x \in C(B)$ if and only if the in-degree of x is zero in the subgraph induced by B denoted $G_{\succ}(B)$.

Thus, in terms of graph theory, Proposition 7 we can either say that an oriented graph is cyclic if and only if it contains some vertex-induced subgraph with no vertex of in-degree zero. Or we can say that a oriented graph is acyclic if and only if every vertex-induced subgraph has a vertex with zero in-degree. These are the same statement. I prefer the second so I will state the proposition in those terms.

Proposition 13. *An oriented (asymmetric) graph* $\{X, \succ\}$ *is acyclic if and only if for every vertex-induced subgraph there is some vertex with zero in-degree.*

The proof of this relies on the three lemmas below.

Lemma 14. G *is acyclic* $\Leftrightarrow$ *Every vertex-induced subgraph of* G *is acyclic.*

Proof. This is almost by definition. We prove the contrapositive that G is cyclic if and only if there is some cyclic vertex induced subgraph. Suppose there is a cycle in some vertex-induced subgraph of G, that cycle also exists in G. Thus, a cyclic subgraph of G implies a cyclic G. Suppose G contains a cycle. Let S be the set of vertices in this cycle. This cycle exists in $G(S)$. Thus, if G is cyclic then there is some cyclic subgraph. □

Lemma 15. *Graph G is acyclic $\Rightarrow G$ has a vertex of zero in-degree.*

This is an elementary result found in many graph theory textbooks. A zero in-degree vertex is called a ***source*** *and a zero out-degree vertex is called a* ***sink***. *Usually the statement of this is something like "every acyclic graph has at least one source and at least one sink". This proof is listed as an exercise below.*

Lemma 16. *Every vertex-induced subgraph of G has a vertex of zero in-degree $\Rightarrow$ G is acyclic.*

I could not find this exact statement anywhere, but it is easy to prove.

Proof. Suppose otherwise, then G is cyclic. It has a cycle $x_1 \succ ... \succ x_n \succ x_1$. Let S be the set of vertices in this cycle. Since every $x \in S$ appears after some $\succ$ in the cycle, every x has a positive in-degree, a contradiction.

Now we can combine these lemmas. The $\Rightarrow$ direction of our proposition is a combination of Lemmas 14 and 15. G is acyclic $\Leftrightarrow$ Every vertex-induced subgraph of G is acyclic $\Rightarrow$ Every vertex-induced subgraph has a vertex of zero in-degree. The $\Leftarrow$ direction is equivalent to Lemma 16. □

3.4 Exercises

3.1 Draw a graph of a $\succsim$ that is complete and quasi-transitive, but not transitive.

3.2 Draw a graph of a $\succsim$ that is complete and acyclic, but not quasi-transitive.

3.3 Prove every (finite) directed acyclic graph has at least one source and at least one sink.

4 From Preference to Utility

The preference relation can be hard to work with. It is almost always easier to work with a numerical summary of $\succsim$ (if one exists) than to work with the relation itself. The utility function is such a summary.

4.1 Utility Represents Preferences

A **utility function** is a mapping from $X \to \mathbb{R}$ such that $U(x) \geq U(x') \Leftrightarrow x \succsim x'$. When such a utility function exists, we say U represents preference relation $\succsim$. When does such a representation exist?

$$\boxed{\textbf{Utility Function } U\text{: } U(x) \geq U(x') \Leftrightarrow x \succsim x'.}$$

4.2 Constructing U under Finite X

For Finite X, Utility Function U Exists $\Leftrightarrow \succsim$ is rational.

I have hinted that there is sometimes an existence problem with U. That's only an issue with uncountable X. For finite and countably infinite X, there is always a utility function as long as $\succsim$ is rational. The following proof relies on this lemma, which is left as an exercise.

Lemma 17. *For rational $\succsim$, $x \succsim y \Leftrightarrow \precsim(y) \subseteq \precsim(x)$*

This is an extension to the proof completed in Exercise 2.10. You will prove this extension in the exercises of this chapter.

Proposition 18. *For finite X, U exists that represents $\succsim \Leftrightarrow \succsim$ is complete and transitive.*

Proof. Let's start with $\Rightarrow$. *If U exists that represents $\succsim$ **then** $\succsim$ is complete and transitive.*

Because $\geq$ is complete on the real numbers, for every $x, y \in X$ either $U(x) \geq U(y)$ or $U(y) \geq U(x)$. Thus, because U represents $\succsim$, $\succsim$ is complete.

By similar argument, $\succsim$ is transitive. For every three $x, y, z \in X$. If $U(x) \geq U(y)$ and $U(y) \geq U(z)$ then $U(x) \geq U(z)$ because $\geq$ is transitive on the real numbers. Since, U represents $\succsim$, $\succsim$ is transitive.

Now we prove $\Leftarrow$. If $\succsim$ is complete and transitive then there is some utility function that represents it. I claim $U(x) = \#(\precsim(x))$ is such a utility function.

To show this, we need to prove: $\#(\precsim(x)) \geq \#(\precsim(y)) \Leftrightarrow x \succsim y$.

Let's start with $x \succsim y \Rightarrow \#(\precsim(x)) \geq \#(\precsim(y))$.

Since preferences are rational, $x \succsim y \Leftrightarrow \precsim(y) \subseteq \precsim(x) \Rightarrow \#(\precsim(x)) \geq \#(\precsim(y))$.

Now we prove $\#(\precsim(x)) \geq \#(\precsim(y)) \Rightarrow x \succsim y$. The contrapositive (by completeness of $\succsim$) of this is: $y \succ x \Rightarrow \#(\precsim(y)) > \#(\precsim(x))$. We prove that instead.

By containment and the fact that indifference curves don't cross, $y \succ x \Leftrightarrow \precsim(x) \subset \precsim(y)$. When X is finite[3], $\precsim(x) \subset \precsim(y) \Rightarrow \#(\precsim(x)) < \#(\precsim(y))$ thus, $y \succ x \Rightarrow \#(\precsim(x)) < \#(\precsim(y))$ ☐

4.3 Constructing U under Countably Infinite X

The only-if part of the proof of Proposition 18 will not work for infinite X, since $\succsim(x)$ may well be infinite. However, it can be modified by picking any arbitrary order on the bundles: $(x_1, x_2, ...)$ and assigning weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences: $u(x) = \sum_{y \in \precsim(x)} w(y)$. With this, we can extend Proposition 18.

Proposition 19. *For countably infinite X, U exists that represents $\succsim \Leftrightarrow \succsim$ is complete and transitive.*

[3]This statement fails with infinite X. For instance, the set of odd numbers O is a subset of the set of primes P $(O \subset P)$. Yet, they have the same cardinality. $\#P = \#O$.

4.4 Failures under Uncountable X

Proposition 20. *There are rational $\succsim$ with no utility representation on an uncountable X.*

Proof. The **lexicographic** preference relation on $\mathbb{R}^2_+$ is a reflexive relation and for any distinct pairs of $(x_1, x_2), (y_1, y_2)$, $(x_1, x_2) \succ (y_1, y_2)$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. $\succsim$ is complete and transitive. Suppose there is a U that represents $\succsim$.

Pick two real numbers $v_2 > v_1$ and construct four bundles $(v_1, 1), (v_2, 1), (v_1, 2), (v_2, 2)$. These bundles are ordered as follows: $(v_2, 2) \succ (v_2, 1) \succ (v_1, 2) \succ (v_1, 1)$. Since there is a utility function representing these preferences, $U(v_2, 1) > U(v_2, 2) > U(v_1, 1) > U(v_1, 2)$. Thus, we can write two disjoint intervals on the reals: $[U(v_2, 1), U(v_2, 2)]$ and $[U(v_1, 1), U(v_1, 2)]$. Because the rationals are dense in the reals, there is a unique rational number in each of these intervals. Since v_1 and v_2 were chosen as arbitrary real numbers, for every real, we can find a unique rational number. That is, we have a mapping from the reals into the rationals. This implies that the cardinality of the rationals is at least as large as that of the reals. $\#\mathbb{Q} \geq \#\mathbb{R}$. This contradicts that the cardinality of the rationals is strictly smaller than the reals. $\square$

4.5 What ensures a utility representation in an uncountable X?

Lexicographic preferences are not problematic on every uncountable X. Suppose cars have horse power in $[0, 999]$ and cup holders take values in $\mathbb{Z}_+$ (non-negative integers). Suppose preferences are lexicographic. Cup holders are more important than horsepower. $u(c_i, h_i) = c_i + \frac{h_i}{1000}$ represents these preferences.

What ensures a representation? In case you are curious about a characterization for uncountable sets, I will leave this here for you to explore.

Proposition 21. *A preference relation is representable by a utility function $U(x) \Leftrightarrow$ There exists countable set X^* such that $\forall x, y \in X$ s.t. $x \succ y$, $\exists x^* \in X^* \subset X$ s.t. $x \succsim x^* \succ y$.*

4.6 Continuous $\succsim$.

Unfortunately, Proposition 21 is hard to work with. We would like to have a condition that is easier to check. There is such a condition, but it is for *continuous* preferences rather than general preferences.

Definition 22. Continuous $\succsim$. A relation $\succsim$ is continuous if and only if $\forall x \in X, \succsim(x)$ & $\precsim(x)$ are closed in X.

Proposition 23. Continuous Representation. *Continuous U exists that represents $\succsim \Leftrightarrow \succsim$ is continuous, complete and transitive.*

4.7 U is Ordinal.

U is a representation of $\succsim$. Because $\succsim$ only provides relative comparisons between pairs of bundles, this is the only meaningful information encoded in U. The fact that one bundle might get two times more utility than another does not mean it is two-times better. We say that the information encoded in U is **ordinal**.

Because U is ordinal, we are free to transform it in a way that maintains the ordinal comparisons.

Proposition 24. ***Monotonic Transformations***. *For any $f : \mathbb{R} \to \mathbb{R}$ that is strictly increasing on the range of U, $f(U)$ represents the same preference relation as $\succsim$.*

Example 25. Transforming Cobb-Douglas preferences. The utility function $x_1^{\alpha} x_2^{\beta}$ represents the same preferences as $\alpha ln(x_1) + \beta ln(x_2)$.

4.8 When is U Cardinal?

U is **cardinal** when the utility directly encodes some quantitative information about underlying preferences. The cardinality is always measured in terms of the underlying goods.

Example 26. Suppose a consumer has a preference relation on $\mathbb{R}_+^2$ and that for any $\forall (x_1, x_2), \exists z > 0 : (x_1, x_2) \sim (z, 0)$. Let $U(x_1, x_2) = z$. Notice that $U(x_1, x_2) = 2U(y_1, y_2)$ now has some meaning: a consumer likes (x_1, x_2) two times more than (y_1, y_1) **when measured in terms of the amount of x_1 the consumer would accept for the bundle.**

We can still transform U from the example above and it will represent the same preferences. However, the fact that it measures preference in terms of amount of x_1 will no longer hold. Transforming the utility function also transforms the encoded cardinal information.

4.9 Exercises

4.1 For rational $\succsim$, $x \succsim y \Leftrightarrow \precsim(y) \subseteq \precsim(x)$ and $x \succ y \Leftrightarrow \precsim(y) \subset \precsim(x)$.

4.3. Show that the lexicographic relation described in the proof of Proposition 20 is not continuous.

4.4. Write a utility representation to extend the example in Section 4.5 to horsepower in $\mathbb{R}_+$.

4.5. Prove Proposition 24.

4.6. Argue that for the utility function $U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ is cardinal with respect to multiples of the bundle $(1, 1)$. Informally, what happens to this encoded information when we take the transformation $ln()$ to get another function that represents the same preferences: $V(x_1, x_2) = \alpha ln(x_1) + (1 - \alpha) ln(x_2) + 1$?

5 Other Properties of $\succsim$.

In this section we discuss some other useful assumptions for $\succsim$.

5.1 Monotonicity.

Definition 27. Weak Monotonicity. $x \geq x' \Rightarrow x \succsim x'$ and if $\forall i$, $x_i > x'_i$ then $x \succ x'$.

Definition 28. Strict Monotonicity. $x \geq x' \Rightarrow x \succsim x'$ and if $x \geq x'$ and $x \neq x'$ then $x \succ x'$.

Definition 29. Local Nonsatiation. $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x^* \in B_\varepsilon(x) \cap X$ such that $x^* \succ x$.

Proposition 30. Monotonicity and U. *If $\succsim$ is locally nonsatiated, then $x \in C(B) \Rightarrow x$ is on the boundary of B.*

Proof. Proof left as an exercise. □

Proposition 31. Monotonicity and U. *If $\succsim$ is represented by U, $\succsim$ weakly (strictly) monotonic $\Leftrightarrow$ U strictly (strongly) increasing.*

Proof. For the definition of *strictly* and *strongly* increasing see Definitions 98, 99 in the appendix. Proof left as an exercise. □

5.2 Convexity.

For more on convexity, see Appendix Section 14.4.

Definition 32. Convex Preferences. $\succsim$ is **convex** iff $x \succsim x' \Rightarrow t(x) + (1-t)x' \succsim x'$, $t \in [0,1]$. It is strictly convex iff $x \succsim x' \Rightarrow t(x) + (1-t)x' \succ x'$, $t \in (0,1)$.

Proposition 33. Convexity of Contours. *$\succsim(x)$ is convex if and only if $\succsim$ is a convex preference relation. $\succsim(x)$ is strictly convex if and only if $\succsim$ is a strictly convex preference relation.*

Proof. Proof left as an exercise. □

Proposition 34. Quasiconvexity of U. *If $\succsim$ is represented by U, then $\succsim$ is (strictly) convex if and only if U is (strictly) quasi-concave.*

Proof. Proof left as an exercise. □

5.3 Homothetic.

Definition 35. Homotheticity. $\forall x, y \in X, \forall t \in \mathbb{R}_+ : x \succsim y \Rightarrow \text{tx} \succsim ty$,.

Proposition 36. Parallel along rays. If u is homothetic and differentiable, $\frac{\frac{\partial u(x)}{\partial x_i}}{\frac{\partial u(x)}{\partial x_j}} = \frac{\frac{\partial u(tx)}{\partial x_i}}{\frac{\partial u(tx)}{\partial x_j}}$.

That is, indifference curves are parallel along rays through the origin.

5.4 Other Assumptions

Here are some additional assumptions that are used in various characterization results. You are not responsible for knowing these, they are just here to give you a taste of what is out there. They are also used in the exercises of this section.

Definition 37. Element-wise Homotheticity. $\forall i \in \{1,...,N\}, \forall t \geq 0$, $(x_1,.,x_i,.,x_n) \succsim (y_1,.,y_i,.,y_n) \Rightarrow (x_1,.,x_i+t,.,x_n) \succsim (y_1,.,y_i+t,.,y_n)$

Definition 38. Substitutability. $\forall x \in X, \exists a \in \mathbb{R}_+$ such that $a(1,...,1) \sim x$

Definition 39. Additivity. $\forall x,y,z \in X$, $x \succsim y \Rightarrow x+z \succsim y+z$

Definition 40. Sensitivity. $\forall i \in \{1,...,n\}$, $\exists x,y \in X$ with $x_j = y_j$ for $j \neq i$ such that $x \not\sim y$.

The following characterizations utilize these axioms. Both of the results can be found in Voorneveld (2008):

Proposition 41. ***Perfect Substitutes Characterization.*** *A complete and transitive preference relation $\succsim$ satisfies strict monotonicity, additivity, and substitutability if and only if there is a utility function of the form $U(x_1,...,x_n) = \sum_{i=1}^{n} \alpha_i x_i$ (with $\alpha_i > 0$) that represents these preferences.*

Proposition 42. ***Cobb Douglass Characterization.*** *A complete and transitive preference relation $\succsim$ satisfies strict monotonicity, element-wise homotheticity, and substitutability if and only if there is a utility function of the form $U(x_1,...,x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}$ (with $\alpha_i \geq 0$) that represents these preferences.*

5.5 Exercises

5.1. Prove proposition 30.

5.2. Prove proposition 31.

5.3. Prove proposition 34.

5.4. Give an intuitive description of a preference relation that does not meet the assumption of *Substitutability.*

5.5. Give an intuitive description of a preference relation that does not meet the assumption of *Additivity.*

5.6. Give an intuitive description of a preference relation that does not meet the assumption of *Sensitivity.*

5.7. Suppose $X = \mathbb{R}^2_+$. Prove or provide a counterexample that the preferences represented by the utility function $min\{x_1, x_2\}$ meet the following assumptions:

> **A.** *Weak Monotonicity*, **B.** *Weak Convexity*, **C.** *Element-wise Homotheticity*,
>
> **D.** *Homotheticity*, **E.** *Substitutability*, **F.** *Additivity*, **G.** *Sensitivity.*

5.8. Suppose $X = \mathbb{R}^2_+$. Prove or provide a counterexample that the preferences represented by the utility function $x_1^{\alpha} x_2^{\beta}$ meet the following assumptions:

> **A.** *Weak Monotonicity*, **B.** *Weak Convexity*, **C.** *Element-wise Homotheticity*,
>
> **D.** *Homotheticity*, **E.** *Substitutability*, **F.** *Additivity*, **G.** *Sensitivity.*

5.6 References

Voorneveld, M. (2008). From preferences to Cobb-Douglas utility (No. 701). SSE/EFI Working Paper Series in Economics and Finance.

6 The Consumer Problem / Constrained Optimization

6.1 The Lagrange Method - Some Intuition.

The purpose of this section is to provide some intuition for the Lagrange method of constrained optimization. A full treatment of constrained optimization will be covered in math camp.

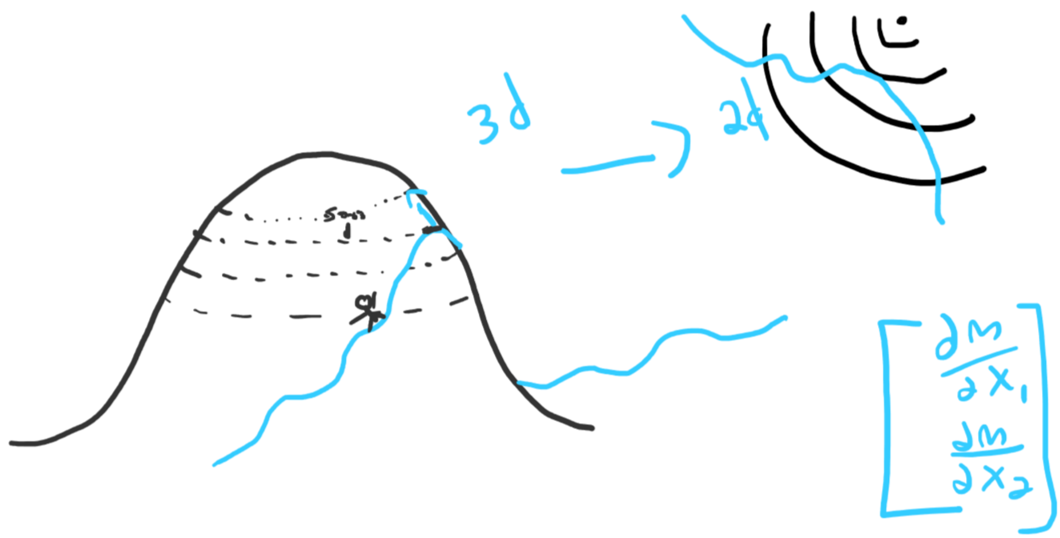

Figure 6.1: Finding the best spot for a selfie.

Key Intuition About Constrained Optimization.
In a constrained optimization problem, suppose both the objective and the constraint are smooth. At the optimal point, **the direction of the gradient of the objective has to be equal to the direction of the gradient of the constraint.** Otherwise, moving along the constraint boundary in *some* direction will yield a larger value of the objective.[a]

[a]This assumes we *can* move in every direction along the constraint. That will only be true at non-boundary points.

Proposition 43. Constrained Optimum Necessary Condition: $\nabla U(x) = \lambda \nabla G(x)$ is necessary for an interior optimum of U constrained by $G(x) = 0$.

Now we create a function for which the first order condition yields the necessary condition: $\mathcal{L} = U(x) - \lambda(G(x))$. This function is called the **Lagrangian**.

6.2 Dual Problem - Some Intuition

When we flip a maximization problem over, we get a minimization problem. Let's flip over the Lagrangian from above. We get: $-\mathcal{L} = -U(x) + \lambda(G(x)) = \lambda(G(x)) - U(x)$. Dividing by λ gives us: $-\frac{1}{\lambda}\mathcal{L} = G(x) - \frac{1}{\lambda}U(x)$. This looks like the Lagrangian of some other kind of constrained problem. That problem is called the **Dual** of the original **Primal** problem.

6.3 The Consumer Problems

Consumer's Constrained Maximization Problem $Max_{x \in X} U(x)\, s.t.\, p \cdot x \leq m$. Is maximized at the **Marshallian Demand** and has optimal value given by the **Indirect Utility Function.**

Consumer's Constrained Minimization Problem $Min_{x \in X}(p \cdot x)\, s.t.\, U(x) \geq u$. Is minimized at the **Hicksian Demand** and has optimal value given by the **Expenditure Function.**

The consumer's goal is to find the set of all "best things" in the budget set. Formally: $C(B) = \{x | x \in B \wedge x \succsim x', \forall x' \in B\}$. Here, we will focus primarily on competitive budgets, but much of what we will do extends to more general budgets.

We can represent the consumer's problem on a competitive budget as the following constrained maximization problem: $Max_{x \in X} U(x)\, s.t.\, p \cdot x \leq m$. This is called the **Consumer's Utility Maximization Problem** The solutions to this problem are called the **Marshallian Demands** $x_i^*(\boldsymbol{p}, y)$ and the optimal value as a function of the parameters is called the **Indirect Utility Function** $V(p, y) = u(x^*(p, y))$.

The dual of this problem is the **Consumer's Cost Minimization Problem**. Its solutions are called the **Hicksian Demands** $x_i^h(\boldsymbol{p}, u)$ and the optimal value as a function of the parameters is called the **Expenditure Function** $e(\boldsymbol{p}, u) = \sum_{i=1}^n p_i x_i^h(\boldsymbol{p}, u)$.

Under conditions on U which we will discuss in class, these problems are **strongly dual**. That is, the solution to one is also the solution to the other in the following sense: if we set the achievable utility given some budget as a constraint and then minimize expenditure for that utility the cost is that original budget and the optimal bundle is the same. In this sense, when our problems are strongly dual, we are free to work on either the dual or primal problem.

Example 44. Suppose a consumer has utility $U(x) = x_1 x_2$. Prices are $p_1 = 1$, $p_2 = 1$ and income is 100. Find the Marshallian demands and achievable utility. Set this utility as a constraint and set up the expenditure minimization problem. Show that the Hicksian demands are the same as the Marshallian demand and the minimized expenditure is 100.

6.4 Properties of Indirect Utility

Proposition 45. *For U that is continuous and strictly increasing, the Indirect Utility Function v has the Following Properties:*

1. Continuous.

This is due to Berge's Maximum Theorem

2. Homogeneous of degree zero in prices and income.

$V(t\boldsymbol{p}\boldsymbol{,}ty) = t^0 V(\boldsymbol{p}, y) = V(p, y)$.

3. Strictly increasing in y and weakly decreasing in p.

4. Quasi-convex in (p, y).

5. Roy's Identity. $-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial m}} = x_i^*$ *(An envelope condition.)* [4]

Exercise 46. Let $B(p, m) = \{x | x \in \mathbb{R}_+^n, px \leq m\}$. Show that $B(tp + (1-t)p', tm + (1-t)m') \subseteq B(p, m) \cup B(p', m')$. Use this to prove that indirect utility is quasi-convex since this is equivalent to $max\{V((p, m)), V((p', m'))\} \geq V((tp + (1-t)p', ty + (1-t)y'))$.

6.5 Properties of Expenditure Function

Proposition 47. *For U that is continuous and strictly increasing, the Expenditure Function e has the following properties:*

1. Continuous.

Berge's Maximum Principle.

2. For $p \gg 0$, strictly increasing and unbounded above in u.

3. Increasing in p.

4. Homogeneous of degree 1 in p.

5. Concave in p.[5]

6. Shephard's lemma. When x_i^h is single valued, $\frac{\partial e}{\partial p_i} = x_i^h$

6.6 Properties of Demand

Slutsky Equation

Slutsky Equation: $\frac{\partial(x_i(p,y))}{\partial p_j} = \frac{\partial\left(x_i^h(p,\bar{u})\right)}{\partial p_j} - \frac{\partial(x_i(p,y))}{\partial y} x_j^h$.

By the duality of utility maximization and expenditure minimization, the Marshallian and Hicksian demand must be equal. That is, $x_i(p, e(p, \bar{u})) = x_i^h(p, \bar{u})$. If we take the derivative of both sides of this equation with respect to a price, we get:

[4]Note: The ratio of the way utility changes with price i to the way it changes with income is proportional to the amount of i consumed. This is because as price of i changes, it changes effective income by $(\Delta p_i) x_i$ and locally, there is no need to worry about changes in consumption level. That is $-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial y}} = -\frac{-x_i^* \lambda}{\lambda} = x_i^*$.

[5]The meaning of this in terms of economics: If x^* is optimal at p, u and prices change. x^* still achieves the utility u. The cost of x^* thus represents an upper bound on the expenditure I need to achieve u at the new prices.

$$\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j} + \frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial y} \frac{\partial e\left(p, \bar{u}\right)}{\partial p_j} = \frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

By Shephard's lemma, $\frac{\partial e(p,\bar{u})}{\partial p_j} = x_j^h$ and so we have:

$$\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j} + \left(\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial y} x_j^h\right) = \frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Let $e\left(p, \bar{u}\right) = y$ and rearrange this to get the **Slutsky Equation.**

$$\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j} = \frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j} - \frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y} x_j^h$$

Negative Own-Substitution Effects

Hicksian demand is decreasing in own-price: $\frac{\partial^2(e)}{(\partial p_i)^2} = \frac{\partial x_i^h}{\partial p_i} \leq 0$.

Proposition 48. Negative Substitution Effect. $\frac{\partial^2(e)}{(\partial p_i)^2} = \frac{\partial x_i^h}{\partial p_i} \leq 0$

Proof. The expenditure function is concave. □

Thus, the substitution effect for a good with respect to its own price must be negative.

Elasticity

Definition 49. Income Elasticity $\eta_i = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial y}{y}} = \frac{\partial x_i}{\partial y} \frac{y}{x_i}$

The income elasticity measures, approximately, the percent change in demand for a one-percent increase in income.

Definition 50. Price and Cross-Price Elasticity $\epsilon_{ij} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_j}{p_j}} = \frac{\partial x_i}{\partial p_j} \frac{p_j}{x_i}$.

The price elasticity measures, approximately, the percent change in demand for a one-percent increase in price.

Elasticity Relations

The share-weighted elasticities with respect to good i is the negative of $i's$ share: $-s_i = \sum_{j=1}^{n} s_j \varepsilon_{j,i}$ **The share-weighted income elasticities sum to 1:** $1 = \sum_{j \in I} s_j \eta_j$

There are some interesting relationships between elasticities that hold when the budget constraint binds. To derive either, start with a statement of the budget equation: $y = \sum_{j \in I} p_j x_j (p, y)$.

Taking the derivative with respect to a price and doing some algebra gets us: $-s_i = \sum_{j=1}^{n} s_j \varepsilon_{j,i}$.

Taking the derivative with respect to income and doing some algebra gets us: $1 = \sum_{j \in I} s_j \eta_j$.

6.7 Exercises

6.1. Let $B(p, m) = \left\{x | x \in \mathbb{R}_+^n, px \leq m\right\}$. Show that $B(tp + (1-t)p', tm + (1-t)m') \subseteq B(p, m) \cup B(p', m')$. Use this to prove that Indirect utility is quasi-convex in (p, m): $max\{V((p, m)), V((p', m'))\} \geq V((tp + (1-t)p', tm + (1-t)m'))$.

6.2. Suppose a consumer has utility $U(x_1 x_2) = \left(\frac{1}{x_1} + \frac{1}{x_2}\right)^{-1}$.

A. Find the consumer's Marshallian Demand.

B. Find the consumer's Hicksian Demand.

6.3. Suppose a consumer has utility $U(x_1 x_2) = x_1^{\alpha} x_2^{\beta}$.

A. Find the consumer's Marshallian Demand.

B. Show that the various properties of the Indirect Utility hold.

C. Show that the various properties of the Expenditure Function hold.

D. Find the price, cross-price, and income elasticities.

**** 6.4.** Every day, a consumer gets a budget set that is **two** randomly chosen bundles from the unit square $X = [0, 1]^2$. The cost of a bundle is $x_1 + x_2$. The consumer buys their favorite bundle.

A. What is the average cost of a randomly chosen bundle?

B. How much does a consumer with utility function $U(x_1, x_2) = x_1 + x_2$ spend on an average day?

C. How much does a consumer with utility function $U(x_1, x_2) = x_1 x_2$ spend on an average day?

7 More Complex Optimization Examples

7.1 Some Examples with Multiple Constraints

Example 51. Maximize $x_1 x_2$ subject to (1) $\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} \leq 10$ and (2) $2x_1 + x_2 \leq 40$.

After plotting the two constraints, we can see that constraint 1 is entirely contained on the interior of constraint 2. The only constraint that could possibly bind is constraint 1. We can ignore constraint 2 and set up the following Lagrangian: $x_1x_2 - \lambda\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}} - 10\right)$. Taking the first order conditions and solving the result gives us the optimal solution: $x_1 = x_2 = \frac{10}{\sqrt{2}}$

Example 52. Maximize x_1x_2 subject to (1) $\left(x_1^2+x_2^2\right)^{\frac{1}{2}} \leq 10$, (2) $2x_1+x_2 \leq 15$ and $x_1, x_2, x_3 \geq 0$.

Now neither constraint is contained in the other. There are several possibilities. Both bind, only (1) binds, only (2) binds.

Suppose both constraints bind. We set up the Lagrangian $x_1x_2-\lambda\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}} - 10\right)-\mu\left(2x_1+x_2-15\right)$.

There is only one point on both boundaries: $x_1 \approx 2.68338, x_2 \approx 9.63325.$, but is cannot be optimal. Let's see how we can tell. The first order conditions of the Lagrangian give us: $x_2 = \frac{\lambda x_1}{\sqrt{x_1^2+x_2^2}} + 2\mu$ and $x_1 = \frac{\lambda x_2}{\sqrt{x_1^2+x_2^2}} + \mu$. Plugging in x_1 and x_2 and solving we get: $\mu \approx 5.16181, \lambda \approx -2.57279$. Notice the negative value of λ. This tells us, to get both to bind, we need to change the direction of constraint (1) from a $\leq$ to a $\geq$ constraint.

To interpret this, the reason both cannot bind is because at the point where both bind, the slope of the boundary of both constraints is shallower than the slope of the indifference curve at that point. We cannot take a linear combination of the constraints and have the slope of that linear combination be equal to the slope of the indifference curve.

Suppose only constraint (1) binds. This situation is the same as in the previous example. The optimum is $x_1 = x_2 = \frac{10}{\sqrt{2}}$. However, at this point, we are not on the interior constraint (2) since $2\left(\frac{10}{\sqrt{2}}\right) + \frac{10}{\sqrt{2}} > 15$.

Suppose only constraint (2) binds. Set up the Lagrangian $x_1x_2 - \mu\left(2x_1+x_2-15\right)$. This has the first order conditions: $x_1 = \mu$, $\frac{x_2}{2} = \mu$. Solving these together with the constraint, we get $x_1 = \frac{15}{4}$ and $x_2 = \frac{30}{4}$ which is the optimal solution.

Example 53. Maximize x_1x_2 subject to (1) $\left(x_1^2+x_2^2\right)^{\frac{1}{2}} \leq 10$, (2) $2x_1+x_2 \leq 20$ and $x_1, x_2, x_3 \geq 0$.

Now neither constraint is contained in the other. There are several possibilities. Both bind, only (1) binds, only (2) binds.

Suppose only constraint (1) binds. This situation is the same as in the previous example. The optimum is $x_1 = x_2 = \frac{10}{\sqrt{2}}$. However, at this point, we are not on the interior constraint (2) since $2\left(\frac{10}{\sqrt{2}}\right) + \frac{10}{\sqrt{2}} > 20$.

Suppose only constraint (2) binds. Set up the Lagrangian $x_1x_2 - \mu\left(2x_1+x_2-15\right)$. This has the first order conditions: $x_1 = \mu$, $\frac{x_2}{2} = \mu$. Solving these together with the constraint, we get $x_1 = 5$ and $x_2 = 10$ which is not on the interior of constraint (1).

Suppose both bind. There are two points on both boundaries: $x_1 = 6, x_2 = 8$ and $x_1 = 10, x_2 = 0$. The second gives zero utility. We can ignore it. The first order conditions of the Lagrangian give us: $x_2 = \frac{\lambda x_1}{\sqrt{x_1^2+x_2^2}} + 2\mu$ and $x_1 = \frac{\lambda x_2}{\sqrt{x_1^2+x_2^2}} + \mu$. Plugging in x_1 and x_2 and solving we get: $\mu = \frac{14}{5}, \lambda = 4$. This is feasible and optimal.

Example 54. Maximize $u = \log(x_1) + \sqrt{x_2} + x_3$ subject to $x_1 + x_2 + x_3 \le m$ and $x_1, x_2, x_3 \ge 0$ Let's set up the Lagrangian function while putting the non-negativity constraints in explicitly this time.

$$\log(x_1) + \sqrt{x_2} + x_3 - \lambda(x_1 + x_2 + x_3 - m) - \mu_1(-x_1) - \mu_2(-x_2) - \mu_3(-x_3)$$

The first order conditions are: $\mu_1 + \frac{1}{x_1} = \lambda$, $\mu_2 + \frac{1}{2\sqrt{x_2}} = \lambda$, $\mu_3 + 1 = \lambda$. Suppose none of our non-negativity constraints bind. Then, by complementary slackness: $\mu_1, \mu_2, \mu_3 = 0$. The first order conditions become $\frac{1}{x_1} = \lambda$, $\frac{1}{2\sqrt{x_2}} = \lambda$, $1 = \lambda$. Solving these for $x_1, x_2, x_3 : 1 = x_1, \frac{1}{4} = x_2, x_3 = m - \frac{5}{4}$. If $m \ge \frac{5}{4}$, this is a valid solution to the problem. However, note that if $m < \frac{5}{4}$, this is not a valid solution since it violates the non-negativity constraint for x_3. What is the optimal solution in that case?

Let's suppose $x_1 \ge 0$ is binding. Thus, $x_1 = 0$. The first order condition on x_1 requires $\frac{1}{x_1} = \lambda - \mu_1$. However, at $x_1 = 0$ this equation cannot hold. Thus, $x_1 > 0$ in any solution. Similarly, we can show that $x_2 \ge 0$ cannot bind since it's first order condition requires $\frac{1}{2\sqrt{x_2}} = \lambda - \mu_2$ which is not true at $x_2 = 0$. The only alternative is that $x_1 > 0$, $x_2 > 0$ and $x_3 = 0$. Since the non-negativity constraints on x_1 and x_2 do not bind, $\mu_1 = \mu_2 = 0$. The first order conditions are: $\frac{1}{x_1} = \lambda, \frac{1}{2\sqrt{x_2}} = \lambda, x_1 = 2\sqrt{x_2}$. Solving these: $x_1 = 2\left(\sqrt{m+1} - 1\right), x_2 = m - 2\sqrt{m+1} + 2$, $x_3 = 0$. Let's look at the multipliers and check that they are positive:$\lambda = \frac{1}{2\left(\sqrt{m+1}-1\right)}, \mu_3 = \frac{1}{2\left(\sqrt{m+1}-1\right)} - 1$. These are positive for any $m < \frac{5}{4}$. Thus, we have the optimal solution for $m < \frac{5}{4}$.

Let's look at $m = 1$ for instance. $\lambda \approx 1.20711, \mu_3 = 0.20711$. The rate at which utility increases if you relax the budget constraint is about 1.20711. Note that when $m > \frac{5}{4}$ this rate is 1. Why is it that the utility increases at a rate less than this if we relax the non-negativity constraint on x_3? Well, if it is relaxed we can decrease x_3 which in-turn relaxes the budget constraint at a rate of one (the price of x_3). This allows us to increase utility at a rate of 1.20711 but decreasing x_3 decreases utility at a rate of 1 so the net effect is 0.20711.

7.2 Exercises

7.1. Maximize $u = x_1 x_2$ subject to $\frac{1}{2}x_1 + 2x_2 \le 100$ and $3x_1 + 2x_2 \le 250$. At the optimum, would the consumer rather relax budget 1 or budget 2?

7.2. Maximize $u = x_1 x_2$ subject to $\frac{1}{2}x_1 + 2x_2 \le 100$ **or** $3x_1 + 2x_2 \le 250$.

7.2. Suppose a consumer has utility function $u = (x_1 + a)^\alpha (x_2 + b)^\beta (x_3 + c)^\gamma$. (assume $a, b, c > 0$)

A. Find the Marshallian demand. Make sure to account for any corner solutions.

B. Find the indirect utility and expenditure functions. Make sure to account for any corner solutions.

7.3. A consumer has income $y = 1$. Price of good 2 is $p_2 = 2$. The consumer's utility function is:

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 & x_1 + x_2 < 1 \\ x_1 + 4x_2 & x_1 + x_2 \ge 1 \end{cases}$$

A. Prove this consumer does not have continuous preferences.

B. Sketch some indifference curves for this consumer. Make sure to include a few above and below the line $x_1 + x_2 = 1$.

C. What is the consumer's Marshaillian demand when $p_1 \in (1, 2)$?

D. What is the consumer's Marshaillian demand when $p_1 \in (0, 1)$?

E. What is unordinary about this demand?

7.4. Solve the following assume throughout that $m > 0$ and $b > 0$.

A. What constraint on m and b make this statement true? $(1, 2) \in \{(x, y) \,|y = -mx + b\}$.

B. Find the m and b meeting the above constraint that minimize the area of the set: $\{(x, y) \,|y \geq 0, x \geq 0, y \leq -mx + b\}$.

C. What constraint on m and b make this statement true? $(1, 2) \in \left\{(x, y) \,|y = -(mx)^2 + b\right\}$.

D. Find the m and b meeting the above constraint that minimize the area of the set: $\left\{(x, y) \,|y \geq 0, x \geq 0, y \leq -(mx)^2 + b\right\}$

8 Decisions Under Uncertainty

Much of the previous sections dealt with bundles that represent *sure* amounts of some goods. This chapter shows how we can work with more complex objects. In this case– uncertain outcomes.

Outcomes: $A \equiv \{a_1, ..., a_n\}$.

We already know how to work with "outcomes". If we wanted to represent a consumers preferences over a finite set of **outcomes** $A \equiv \{a_1, ..., a_n\}$ there would be no real difficulty. However, we want the objects here to be uncertainty over those outcomes. We construct these objects first by defining **simple gambles** which are a probability distribution over outcomes. Formally, the set of simple gambles is $\mathcal{G}_s \equiv \{(p_1 \circ a_1, p_2 \circ a_2, ..., p_n \circ a_n) \,|\sum p_i = 1, p_i \geq 0\}$.

Set of Simple Gambles: $\mathcal{G}_s \equiv \{(p_1 \circ a_1, p_2 \circ a_2, ..., p_n \circ a_n) \,|\sum p_i = 1, p_i \geq 0\}$.

Generically simple gambles are denoted with g. For instance, we might write $g_1, ..., g_n \in \mathcal{G}_s$. We now turn it up a notch and define the set of **First-order Compound Gambles.** A first-order compound gamble is a simple lottery where the outcomes are simple gambles. Formally, $\mathcal{G}_{c_1} \equiv \{(p_1 \circ g_1, p_2 \circ g_2, ..., p_n \circ g_n) \,|\sum p_i = 1, (\forall i \in \{1, ..., n\}, g_i \in \mathcal{G}_s)\}$.

Set of First-Order Compound Gambles:

$$\mathcal{G}_{c_1} \equiv \left\{(p_1 \circ g_1, p_2 \circ g_2, ..., p_n \circ g_n) \,|\sum p_i = 1, p_i \geq 0, (\forall i \in \{1, ..., n\}, g_i \in \mathcal{G}_s)\right\}$$

We can now define gambles of arbitrary complexity by recursion. Let $\mathcal{G}_{C_j}$ be the set of $j - th$ order compound gambles. The set of all compound gambles are:

Set of All Compound Gambles:

$$\mathcal{G} \equiv \left\{ (p_1 \circ g_1, p_2 \circ g_2, ..., p_m \circ g_m) \mid \sum p_i = 1, p_i \geq 0, \left(\exists j \in \mathbb{N}, g_i \in \mathcal{G}_{c_j}\right) \right\}$$

Example 55. Compound Gamble. Let the set of outcomes be $A = \{\$10, \$7, \$5, \$0\}$. $g_1 = \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right)$ is a simple gamble. $g_2 = (1 \circ \$5)$ is a simple (degenerate) gamble. $g_3 = \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ (1 \circ \$5)\right)$ is a compound gamble consisting of two simple gambles. $g_4 = \left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right), \left(\frac{1}{2} \circ \$7\right)\right)$ is a compound gamble consisting of a compound gamble that consists of two simple gambles (one degenerate), and a simple degenerate gamble. *Oof.*

8.1 Induced Simple Gambles and Reduction

Since every gamble is based on the building blocks of outcomes, we can take a compound gamble and calculate it's implied distribution over the outcomes. This is the **induced simple gamble** or the **simple gamble induced by** g**.**

Example 56. Induced Simple Gamble. Consider the compound gamble $g = \left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right), \left(\frac{1}{2}\right.\right.$ The induced simple gamble is $g_s = \frac{1}{8} \circ \$10, \frac{1}{8} \circ \$0, \frac{1}{4} \circ \$5, \frac{1}{2} \circ \$7$.

You might think that consumers would always be indifferent between g and g_s in the example above. They are two gambles that result in the exact same chance of every outcome. The only thing that is different between them is the random process that determines the outcome. However, if we define preferences over $\mathcal{G}$, there is no reason we *need* to make that the case.

Let's define $g_s(g)$ as the function that maps any gamble g into it's induced simple gamble. We can write the assumption that would give us that succinctly.

Definition 57. ***Reduction.*** $\forall g \in \mathcal{G}, g \sim g_s(g)$.

8.2 Expected Utility

If we assume reduction, you might think we can represents a consumer's preferences by associating a utility with each outcome and then simply taking the average utility they get using the probabilities of each outcome from the induced simple gamble.

In Example 56, the induced simple gamble was $g = \left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right), \left(\frac{1}{2} \circ \$7\right)\right)$ and the induced simple gamble is $g_s = \frac{1}{8} \circ \$10, \frac{1}{8} \circ \$0, \frac{1}{4} \circ \$5, \frac{1}{2} \circ \$7$. If we had utilities associated with each outcome $u(\$10), u(\$7), u(\$5), u(0)$ it would be tempting to say that the consumer's utility for $u(g) = \frac{1}{8}u(\$10) + \frac{1}{8}u(\$0) + \frac{1}{4}u(\$5) + \frac{1}{2}u(\$7)$. If it were always the case, we can write utility as follows. Let $p_i(g)$ be the probability of outcome i in the simple lottery induced by g. Let $E_g(u(a_i)) = \sum_{i=1}^{n} p_i(a_i) u(a_i)$. When this is the case, we say the preferences have the expected utility property.

Definition 58. Expected Utility Property. $\forall g \in \mathcal{G}, U(g) = E_g(u(a_i))$**.**

If U represents $\succsim$ and has the expected utility property we call it an expected utility function. Not all preferences over $\mathcal{G}$ have an expected utility function that represents those preferences.

Example 59. Linear Utility. Consider the compound gamble from Example 56. $g = \left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right.\right.\right.$ Suppose utility over money is linear so that $u(x) = x$ and that the consumer's preferences can be represented by an expected utility function. $u(g) = \frac{1}{8}(10) + \frac{1}{8}0 + \frac{1}{4}5 + \frac{1}{2}7 = 6$

Note that in the above case, the consumer's utility for the gamble g is the same as their utility for the expected outcome of the gamble which is \$6. We call such a consumer risk-neutral. We will look more at this later.

Example 60. Log Utility. Consider the compound gamble from Example 56. $g = \left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right),\right.$ Suppose utility over money is logarithmic in the amount so $u(x) = ln(x+1)$ and that the consumer's preferences can be represented by an expected utility function. $u(g) = \frac{1}{8}(ln(10+1)) + \frac{1}{8}(ln(0+1)) + \frac{1}{4}(ln(5+1)) + \frac{1}{4}(ln(7+1)) = 1.26754$.

Notice here that the utility of the gamble is lower than the utility of the expected outcome (still \$6) since $u(6) = ln(6+1) = 1.94591$. When this happens we say the consumer is risk-averse. Please draw your attention to how the concavity of the utility for money creates this risk-aversion. We will come back to this later.

You might wonder now, "**When can we represent preferences over $\mathcal{G}$ with an expected utility function?**" Let's see.

8.3 Expected Utility Theorem

Let $\succsim$ be the preference relation on $\mathcal{G}$:

Axiom 1. ***Completeness.*** *$\succsim$ is complete.*

Axiom 2. ***Transitivity.*** *$\succsim$ is transitive.*

Assume $a_1 \succ a_2 ... \succ a_n$.

Axiom 3. ***Monotonicity.*** For all $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ iff $\alpha \geq \beta$,

Axiom 4. ***Continuity.*** For all g $\exists p \in [0,1]$ such that $g \sim (p \circ a_1, (1-p) \circ a_n)$

Proposition 61. ***Utility Representation.*** *Under axioms $1-4$, we can represent $\succsim$ with a utility function.*

Proof. Construct $u(g)$ as follows, find the simple gamble over the best and worst outcome that g is indifferent to. Let p be the probability of the best outcome in that simple gamble. Then $u(g) = p$. For instance, if $g \sim \left(\frac{1}{4} \circ a_1, \frac{3}{4} \circ a_n\right)$ then $u(g) = \frac{1}{4}$. Implicitly, we can define $u(g)$ as: $u(g) : g \sim (u(g) \circ a_1, (1-u(g)) \circ a_n)$.

Let's see that $u(g) \geq u(g') \Leftrightarrow g \succsim g'$. Start with $g \succsim g'$. By **continuity**, these are indifferent to gambles over the best and worst outcome: $g \sim (p \circ a_1, (1-p) \circ a_n)$ and $g' \sim (p' \circ a_1, (1-p') \circ a_n)$. Thus by **transitivity** $g \succsim g' \Leftrightarrow (p \circ a_1, (1-p) \circ a_n) \succsim (p' \circ a_1, (1-p') \circ a_n)$.

By **monotonicity** the right side if true if and only if $p \geq p'$. We now have: $g \succsim g' \Leftrightarrow p \geq p'$. By construction, $u(g) = p$. Thus: $g \succsim g' u(g) \geq u(g')$ $\square$

8.4 Expanded Proof- Linearity

In the above utility construction we measure utility through the probabilities of the indifferent gambles over the best and worst outcome. But, under the above definition, the simple gamble $(p \circ a_1, (1-p) \circ a_n)$ that is indifferent to g *can be anything.* However, for a utility function with the expected utility property, $u(g) = u(g_s)$ since all that matters are the utility of the outcomes and the probabilities of those outcome. However, if we have enough structure to ensure that simple lottery over the best and worst outcomes that is indifferent to any gamble g is the same simple lottery over the best and worst outcomes that is indifferent to the simple lottery induced by g, then we can extend the above construction and show that it is linear.

We need two additional assumptions:

Axiom 5. ***Substitutibility.*** For $g = (p_1 \circ g_1, ..., p_k \circ g_k)$ and $h = (p_1 \circ h_1, ..., p_k \circ h_k)$, $g_i \sim h_i$ for all $i \in \{1, ..., k\}$ implies $g \sim h$.

Axiom 6. ***Reduction.*** For any gamble g and the simple gamble it induces g_s, $g \sim g_s$.

Now, construct utility this way: let $u(a_i)$ be defined as above: $u(a_i) : a_i \sim (u(a_i) \circ a_1, (1 - u(a_i)) \circ a_n)$

Note, instead of finding the simple gamble over the best and worst outcome for every g**.** Here, it is enough to know only those indifferent lotteries *for outcomes.* Then we extend the utility function to all gambles through expectation. Let p_i^g be the probability of outcome a_i in the simple gamble induced by g. $u(g) = \sum_{i=1}^n p_i^g u(a_i)$.

We have the following result,

Theorem 62. ***Expected Utility Representation.*** *There is a u with the* expected *utility property such that $u(g) \geq u(g) \Leftrightarrow g \succsim g$ if and only if $\succsim$ meets axioms 1-6.*

Lemma 63. *Let $a \sim b$ and $c \sim d$. If $\succsim$ is transitive, $b \succsim c$ if and only if $a \succsim d$.* ***Proof left as exercise.***

Proof. By ***Reduction***: $g \sim (p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n,)$. By ***Continuity*** every outcome is indifferent to some simple gamble over the best and worst outcome: $a_i \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$. *By* ***Substitution*** we can glue these simple gambles in to the above expression to eliminate all events except a_1 and a_n and turn the compound gamble over many outcomes into a *compound gamble over only the best and worst outcome.* We have $g \sim (p_1^{g_s} \circ a_1, ..., p_i^{g_s} \circ (\alpha_i \circ a_1, (1-\alpha_i) \circ a_n), ..., p_n^{g_s} \circ a_n)$. We can now apply ***Reduction*** a second time to make this a *simple lottery over the best and worst outcome*: $g \sim (\sum_{i=1}^n \alpha_i p_i^{g_s} \circ a_1, \sum_{i=1}^n (1-\alpha_i) p_i^{g_s} \circ a_n)$. Similarly $g' \sim \left(\sum_{i=1}^n \alpha_i p_i^{g_s'} \circ a_1, \sum_{i=1}^n (1-\alpha_i) p_i^{g_s'} \circ a_n \right)$.

By Lemma 63, $g \succsim g' \Leftrightarrow (\sum_{i=1}^n \alpha_i p_i^{g_s} \circ a_1, \sum_{i=1}^n (1-\alpha_i) p_i^{g_s} \circ a_n) \succsim \left(\sum_{i=1}^n \alpha_i p_i^{g_s'} \circ a_1, \sum_{i=1}^n (1-\alpha_i) p_i^{g_s'} \circ a_n \right)$**.** We prove the right side of this.

By **Monotonicity**, this preference is true if and only if $\sum_{i=1}^n \alpha_i p_i^{g_s} \geq \sum_{i=1}^n \alpha_i p_i^{g_s'}$. By construction of $u(g)$: $u(a_i) = \alpha_i$. So this is true if and only if: $\sum_{i=1}^n u(a_i) p_i^{g_s} \geq \sum_{i=1}^n u(a_i) p_i^{g_s'}$. This is precisely $u(g)$ and $u(g')$ so we have that:

$$\left(\sum_{i=1}^{n}\alpha_i p_i^{g_s}\circ a_1,\sum_{i=1}^{n}(1-\alpha_i)\,p_i^{g_s}\circ a_n\right)\succsim\left(\sum_{i=1}^{n}\alpha_i p_i^{g_s^{'}}\circ a_1,\sum_{i=1}^{n}(1-\alpha_i)\,p_i^{g_s^{'}}\circ a_n\right)\Leftrightarrow u(g)\geq u(g')$$

Thus, $g \succsim g'$ if and only if $u(g) \geq u(g')$. □

8.5 Risk Preferences

In Section 8.2 we saw some examples of gambles over wealth. We saw an instance where the utility of the gamble was the same as the utility of the expected outcome of the gamble. In this case, we said the consumer was risk-neutral. We also saw an instance where the utility of the gamble was lower than the utility of the expected outcome of the gamble. In this case, we said the consumer was risk-averse. We can formalize these as follows. Let $E_g(a_i)$ be the expected outcome of the gamble.

Risk Preferences:
Risk Averse: $E_g(u(a_i)) < v(E_g(a_i))$
Risk Loving: $E_g(u(a_i)) > v(E_g(a_i))$
Risk Neutral: $E_g(u(a_i)) = v(E_g(a_i))$

Theorem 64. ***Jensen's Inequality.*** For a all random variables X $E_X(f(x)) \leq (\geq) f(E_X(x))$ if and only if f is concave (convex).

Mapping this into gambles, a consumer is risk-averse if and only if their utility for money is concave. A consumer is risk-loving if and only if their utility for money is convex. A consumer is risk-neutral if and only if their utility for money is linear.

8.6 Certainty Equivalent / Risk Premium

For risk-averse and risk-loving consumers, their utility for the expected outcome of a gamble is not the same at the utility of the gamble. We might ask, "**what expected amount of money would make them indifferent?"**. That's the **certainty equivalent.** Let u be the utility function for outcomes (amounts of wealth) and U the utility function for gambles:

Certainty-Equivalent: *Certainty equivalent* $c \in \mathbb{R}_+$ is the c that solves $u(c) = U(g)$

The **risk premium** is the difference between the expected outcome of a gamble and the certainty equivalent. This represents how much money the consumer is willing to give up in expectation to avoid uncertainty. It is a positive number when the consumer is risk averse.

Risk Premium: *Risk premium* r: $r = E_g(a_i) - c$

Exercise 65. The consumer's preferences for gambles can be represented by an expected utility function with utility for wealth given by $u(x) = ln(x)$. Their starting wealth is w and owns a gamble pays \$0 and \$1000 both with $\frac{1}{2}$ chance so the consumer will have $\left(\frac{1}{2}\circ\$(w),\frac{1}{2}\circ\$(w+1000)\right)$. Calculate the certainty equivalent and risk premium for this gamble as a function of w. Plot the risk premium as a function of w. Give an intuitive explanation for what we are seeing.

8.7 Two Measures of Risk Preferences and Differential Equations

Some consumers are more risk-averse than others. There are a few measures you should be aware of.

Absolute Risk Aversion: $-\frac{v''(w)}{v'(w)}$ **Relative Risk Aversion**:$-\frac{wv''(w)}{v'(w)}$

Both of these are reasonable measures of the curvature of a utility function and have some nice properties. I will not discuss them much, but I bring them up to demonstrate a useful technique for finding functions that have properties given in terms of derivatives. Suppose we want to use a utility function with *constant relative risk aversion* for some model. What utility functions can we use? Formally, we want to find all utility functions such that: $-\frac{wv''(w)}{v'(w)} = c$. This differential equation has solutions $v(w) = \frac{c_1 w^{1-c}}{1-c} + c_2$. Every utility function for wealth with constant relative risk aversion is an affine transformation of $v(w) = \frac{w^{1-c}}{1-c}$.

Exercise 66. Find the form of utility functions for wealth that have constant absolute risk aversion by setting up and solving a differential equation.

8.8 Non-Expected Utility Preferences

The axioms presented above are reasonable, but they are strong. Consider the following pair of lotteries:

$$g_1 = \frac{1}{3} \circ (0.75 \circ \$10, 0.25 \circ \$0), \frac{2}{3} \circ (0.75 \circ \$10, 0.25 \circ \$0)$$

$$g_2 = \frac{1}{3} \circ (0.56 \circ \$10, 0.44 \circ \$0), \frac{2}{3} \circ (0.89 \circ \$10, 0.11 \circ \$0)$$

If we were to reduce both of these we get:

$$g_s(g_1) = 0.75 \circ \$10, 0.25 \circ \$0$$

$$g_s(g_2) = 0.78 \circ \$10, 0.22 \circ \$0$$

Under **reduction** and **monotonicity,** $g_1 \succ g_2$, yet in experiments I ran with Kristine Koutout and Andrew Dustan in 2021, over $\frac{2}{3}$ of participants chose g_2. To be fair, this is not overwhelming evidence against reduction. It is just one pair of lotteries that are pretty close to each other. Maybe close enough that subjects just do not care. Still, there's something attractive about g_1 that is hard to deny and it is easy to cook up preferences that would be consistent with this choice.

Example 67. Lexicographic Minimax Preferences over Uncertainty. Suppose $A = \{a_1, ..., a_n\}$ (ordered such that $a_1 \succ a_2 ... \succ a_n$) let $p^i_{min}(g)$ be the minimum probability of a_i in any simple lottery in g. Define $\succsim$ to be $g_1 \succ g_2$ if $\exists i \in \{1, ..., n-1\}$ such that $p^i_{min}(g_1) > p^i_{min}(g_2)$ and for all $j \in \{1, ..., i-1\}$, $p^j_{min}(g_1) = p^j_{min}(g_2)$ or if $\forall i \in \{1, ..., n-1\}$ $p^i_{min}(g_1) = p^i_{min}(g_2)$ and $p^n_{min}(g_1) < p^n_{min}(g_2)$.

Note the final condition implies that $g_1 \succ g_2$ for the following two lotteries:

$$g_1 = \frac{1}{3} \circ (0.5 \circ \$10, 0.25 \circ \$5, 0.25 \circ \$0), \frac{1}{3} \circ (0.25 \circ \$10, 0.375 \circ \$5, 0.375 \circ \$0), \frac{1}{3} \circ (0.5 \circ \$10, 0.5 \circ \$5, 0 \circ \$0)$$

$$g_2 = \frac{1}{3} \circ (0.5 \circ \$10, 0.25 \circ \$5, 0.25 \circ \$0), \frac{1}{3} \circ (0.25 \circ \$10, 0.5 \circ \$5, 0.25 \circ \$0), \frac{1}{3} \circ (0.25 \circ \$10, 0.25 \circ \$5, 0.5 \circ \$0)$$

To see that $g_1 \succ g_2$:

$$p^1_{min}(g_1) = 0.25, p^2_{min}(g_1) = 0.25, p^3_{min}(g_1) = 0.0$$

$$p^1_{min}(g_2) = 0.25, p^2_{min}(g_2) = 0.25, p^3_{min}(g_2) = 0.25$$

Exercise 68. Prove whether the preferences *Example 67* in the above example meets is completeness (on $\mathcal{G}$), transitivity, monotonicity, continuity, substitutibility, reduction.

8.9 Exercises

8.1. Suppose a consumer has utility for wealth of $u(w) = w^\alpha$ and is an expected utility maximizer. Their initial wealth is 1000. There is a $\frac{1}{2}$ chance an event that will lead to a 500 loss. Calculate the consumer's willingness to pay for insurance that guarantees no loss if the event happens. Compare the willingness to pay for consumers with $\alpha \in \left\{2, 1, \frac{1}{2}, \frac{1}{4}\right\}$.

8.2. Look up the Saint Petersburg paradox on wikipdia. Suppose a consumer has an initial wealth of \$0 and utility for wealth of $\sqrt{w}$. Calculate their certainty equivalent for the gamble involved in the paradox. Feel free to use any tool you need to solve this problem.

8.3. Suppose a consumer has utility for wealth of $u(w) = ln(w)$ and is an expected utility maximizer. Their starting wealth is w. Calculate their risk premium of a gamble that pays \$1000 or \$0 both with $\frac{1}{2}$ chance (on top of their initial wealth). Plot the risk premium as a function of w. Give an intuitive explanation for what we are seeing.

8.4. Find the form of utility functions for wealth that have constant absolute risk aversion by setting up and solving a differential equation.

8.6. Prove whether or not the preferences *Example 67* meet each of the axioms of completeness, transitivity, monotonicity, continuity, substitutibility, reduction.

8.7. Find a preference relation that is meets completeness (on $\mathcal{G}$), transitivity, monotonicity, continuity, but not substitutibility or reduction.

9 (Optional) Beliefs

9.1 Events and mixed lotteries.

The previous section considered lotteries where the probabilities are known. We can extend such an environment to "events" with unknown probability. The goal of this section is not to go through a full treatment of this environment, but rather to set up just enough to understand how *beliefs* can be elicited. Where Ω is the set of states of the world.

Events Space: $E = 2^{\Omega}$

Recall that a simple lottery of the form $p_1 \circ a_1, p_2 \circ a_2, ..., p_n \circ a_n$. We can similarly define a simple *subjective* lottery.

Set of Simple Subjective Gambles: $\mathcal{H}_s \equiv \{(e_1 \circ a_1, ..., e_n \circ a_n) \,|\, \cup_{i=1}^{n} e_i = \Omega \,\&\, e_i \cap e_j = \emptyset \, for \, i \neq j\}$.

In the previous chapter, we proceeded to define the first-order compound lotteries as probability distributions over simple gambles. We can do the same here, but expand the types of simple gambles that can appear to include subjective ones. Doing this, we get what I call the first-order objective mixtures.

We can likewise define more complex gambles that involve subjective events that result in either a simple subjective or simple objective gamble. I call these *first-order compound mixtures.*

Set of First-Order Objective Mixtures: $\mathcal{H}_{o_1} \equiv \{(p_1 \circ m_1, ..., p_n \circ m_n) \,|\, \cup_{i=1}^{n} e_i = \Omega \,\&\, e_i \cap e_j = \emptyset \, for \, i \neq j, m_i \in \{\mathcal{G}_s, \mathcal{H}_s\}\}$

However, notice we could likewise have subjective events replace the objective probability distribution and get a type of gamble I call a first-order subjective mixture. Note that the set of purely objective first-order compound gambles is a subset of the set of first-order objective mixtures $\mathcal{G}_{c_1} \subset \mathcal{H}_{o_1}$.

Set of First-Order Subjective Mixtures: $\mathcal{H}_{s_1} \equiv \{(e_1 \circ m_1, ..., e_n \circ m_n) \,|\, \cup_{i=1}^{n} e_i = \Omega \,\&\, e_i \cap e_j = \emptyset \, for \, i \neq j, m_i \in \{\mathcal{G}_s, \mathcal{H}_s\}\}$

From here, we could continue to descend into ever-more-complex types of gambles. However, for the purposes of this chapter, stopping at first-order mixtures is enough.

Set of First-Order Mixtures: $\mathcal{H}_1 \equiv \mathcal{H}_{o_1} \cup \mathcal{H}_{s_1}$

For the rest of this chapter, let's assume agents have a preference relation $\succsim$ on $\mathcal{H}_1$. We will extend a few of the axioms from the objective gamble section here.

First, let's make this preference relation rational, monotone, and continuous:

Axiom 1. ***Completeness.*** *$\succsim$ is complete.*

Axiom 2. ***Transitivity.*** *$\succsim$ is transitive.*

Assume $a_1 \succ a_2 ... \succ a_n$.

Axiom 3. ***Monotonicity.*** For all $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ iff $\alpha \geq \beta$

Axiom 4. ***Continuity.*** For all $g \in \mathcal{H}_1$ $\exists p \in [0,1]$ such that $g \sim (p \circ a_1, (1-p) \circ a_n)$

These axioms would be enough to define a utility function over $\mathcal{H}_1$.

9.2 Beliefs

Let's not hassle with philosophy and stick to a simple definition of beliefs:

Belief: A belief about an event e is a p that makes the following statement true.

$$\forall a_1, a_2 \in A \, s.t. \, a_1 \neq a_2, (e \circ a_1, e \circ a_2) \sim (p \circ a_1, (1-p) \circ a_2)$$

Notice that generically, such a p might not exist. The continuity axioms only ensures there is some p that exists for any $(e \circ a_1, e \circ a_2)$ but does not ensure that the p creating this indifference is the same for $(e \circ a_1, e \circ a_2)$ and $(e \circ a_3, e \circ a_4)$. That is, the probability depends on the particular outcomes chosen. If that's the case, finding beliefs will be particularly difficult using choice data. On the other hand, if we assume that the p is independent of a_1, a_2 then we have something to look for. Let's strengthen continuity with the following axiom:

Axiom 7. ***Coherent Beliefs.*** $\forall e \in E, \exists p \in [0,1]$ $s.t. \forall a_1, a_2 \in A \, s.t \, a_1 \neq a_2, (e \circ a_1, e' \circ a_2) \sim (p \circ a_1, (1-p) \circ a_2)$

This axiom is enough to get us started on eliciting beliefs. However, to make use of this definition of beliefs within more complex gambles, we will need to extend the substitution axiom from the previous chapter to apply to these new types of lotteries.

Axiom 5. ***Substitutibility on Mixtures.*** *If $(e \circ a_1, e' \circ a_2) \sim (p \circ a_1, (1-p) \circ a_2)$ for all $a_1, a_2 \in A$, then $(e \circ m_1, e' \circ m_2) \sim (p \circ m_1, (1-p) \circ m_2)$ for any $m_1, m_2 \in \{\mathcal{G}_s, \mathcal{H}_s\}$*

9.3 The Binarized Quadratic Scoring Rule

Definition 69. The **Binarized Quadratic Scoring Rule** is a tool for eliciting a belief (as defined here). It is a family of lotteries $H(\tilde{p}) = (e \circ g(\tilde{p}), e^c \circ g(1-\tilde{p}))$ where $g(\tilde{p}) = \left(1 - (1-\tilde{p})^2\right) \circ a_1, (1-\tilde{p})^2 \circ a_2$ with $a_1 \succ a_2$.

The procedure asks subjects their belief about the probability of event e. If the say their belief is $\tilde{p}$ they are paid with the lottery $H(\tilde{p})$ from *Definition* 69. This procedure will elicit the subject's true belief p (as defined above) as long as they meet one final assumption– reduction on purely objective lotteries.

Axiom 6. ***Reduction.*** For any gamble $g \in \mathcal{G}_{c_1}$ and the simple gamble it induces g_s, $g \sim g_s$.

To summarize the axioms presented in this chapter so far:

Axioms for BQSR

Axiom 1. ***Completeness.*** $\succsim$ *is complete.*

Axiom 2. ***Transitivity.*** $\succsim$ *is transitive.*

Assume $a_1 \succ a_2 ... \succ a_n$.

Axiom 3. ***Monotonicity.*** For all $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ iff $\alpha \geq \beta$

Axiom 4. ***Continuity.*** For all $g \in \mathcal{H}_1$ $\exists p \in [0,1]$ such that $g \sim (p \circ a_1, (1-p) \circ a_n)$

Axiom 5. ***Substitutibility on Mixtures.*** *If* $(e \circ a_1, e' \circ a_2) \sim (p \circ a_1, (1-p) \circ a_2)$ *for all* $a_1, a_2 \in A$, *then* $(e \circ m_1, e' \circ m_2) \sim (p \circ m_1, (1-p) \circ m_2)$ *for any* $m_1, m_2 \in \{\mathcal{G}_s, \mathcal{H}_s\}$

Axiom 6. ***Reduction.*** For any first-order objective mixture $g \in \mathcal{H}_{o_1}$ and the simple gamble it induces g_s, $g \sim g_s$.

Axiom 7. ***Coherent Beliefs.*** $\forall e \in E, \exists p \in [0,1]\ s.t. \forall a_1, a_2 \in A\ s.t\ a_1 \neq a_2, (e \circ a_1, e' \circ a_2) \sim (p \circ a_1, (1-p) \circ a_2)$

Example 70. Will it rain?

Suppose someone believes the event e ("it will rain tomorrow") is 0.75 and meets all of the axioms defined in this chapter.

If they say that the probability is 0.75, the lottery they receive is:

$$H(\tilde{p}) = (e \circ (0.9375 \circ a_1, 0.625 \circ a_2), e^c \circ (0.4375 \circ a_1, 0.5625 \circ a_2))$$

Notice that by the coherent belief axiom and our new substitution axiom, this lottery is indifferent to the purely objective lottery:

$$0.75 \circ (0.9375 \circ a_1, 0.625 \circ a_2), 0.25 \circ (0.4375 \circ a_1, 0.5625 \circ a_2)$$

We can then apply reduction to get the simple lottery:

$$0.8125 \circ a_1, 0.1875 \circ a_2$$

On the other hand if they said the probability was 0.5, the lottery they get is indifferent to:

$$0.75 \circ a_1, 0.25 \circ a_2$$

By monotonicity, telling the truth (that the belief is 0.75) is better than claiming their belief is 0.5.

Exercise 71. Prove under axioms $1-6$ and 7 (in this chapter), the BQSR incentive compatible. That is for a subject with belief p about event e, $H(p) \succ H(\tilde{p})$ for all $\tilde{p} \neq p$.

9.4 A Simpler, Coarse Alternative

The BQSR elicits a belief in a continuous way. That is, it identifies and exact p. If you are only interested in more coarse categorization of beliefs, there is a simpler alternative.

A **coarse belief menu** is a set of $\{p_1, p_2, ..., p_n\}$ such that $p_i > 0.5$ and $p_1 > p_2 > ... > p_n$, and outcomes $a_1 \succ a_2$. Construct the following menus of gambles.

$$\{(e \circ a_1, e^c \circ a_2), (e^c \circ a_1, e \circ a_2), (p_i \circ a_1, (1 - p_i \circ a_2))\},$$

$$...$$

$$\{(e \circ a_1, e^c \circ a_2), (e^c \circ a_1, e \circ a_2), (p_n \circ a_1, (1 - p_n \circ a_2))\}$$

Have subjects choose one gamble from each menu. Randomly pick a menu (with probability $\frac{1}{n}$) and pay the subject with their chosen gamble from that menu.

Suppose subjects meet the following axioms:

Axioms for Coarse Elicitation

> ***Completeness.*** *$\succsim$ is complete.*
>
> ***Transitivity.*** *$\succsim$ is transitive.*
>
> ***Monotonicity.*** For all $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ iff $\alpha \geq \beta$
>
> ***Coherent Beliefs.*** $\forall e \in E, \exists p \in [0,1]\ s.t. \forall a_1, a_2 \in A\, s.t\, a_1 \neq a_2, (e \circ a_1, e' \circ a_2) \sim (p \circ a_1, (1-p) \circ a_2)$
>
> ***Consistent Beliefs.*** $(e \circ a_1, e' \circ a_2) \sim (p \circ a_1, (1-p) \circ a_2) \Leftrightarrow (e' \circ a_1, e \circ a_2) \sim ((1-p) \circ a_1, p \circ a_2)$
>
> ***Statewise Monotonicity.*** For any gamble $g \in \mathcal{H}_{o_1}$, $g = (p_1 \circ m_1, ..., p_n \circ m_n)$ if m_i is replaced with $m_i^{'} \succ m_i$ to make $g^{'}$ then $g' \succ g$.

Proposition 72. *Under the above axioms, a choices from a coarse elicitation menus with probabilities $\{p_1, ..., p_n\}$ categorize beliefs into the sets:*

$$[0, 1-p_1], [1-p_1, 1-p_2], ... [1-p_{n-1}, 1-p_n], [1-p_n, p_n], [p_n, p_{n-1}] ..., [p_2, p_1], [p_1, 1]$$

For example, using the following two menus will categorize beliefs into categorizes: $[0, 0.2], [0.2, 0.4], [0.4, 0.6], [0.6, 0.8], [0.$

$$\{(e \circ a_1, e^c \circ a_2), (e^c \circ a_1, e \circ a_2), (0.8 \circ a_1, 0.2 \circ a_2)\}$$

$$\{(e \circ a_1, e^c \circ a_2), (e^c \circ a_1, e \circ a_2), (0.6 \circ a_1, 0.4 \circ a_2)\}$$

9.5 Exercises

9.1. Prove under axioms $1-6$ and 7 (in this chapter), the BQSR incentive compatible. That is for a subject with belief p about event e, $H(p) \succ H(\tilde{p})$ for all $\tilde{p} \neq p$.

9.2. Show that under the axioms for coarse elicitation the choice from the menus:
$\{(e \circ a_1, e^c \circ a_2), (e^c \circ a_1, e \circ a_2), (p \circ a_1, (1-p \circ a_2))\}$ and $\{(e \circ a_1, e^c \circ a_2), (e^c \circ a_1, e \circ a_2)\}$
can be used to categorize beliefs into $[0, 1-p], [1-p, 0.5], [0.5, p], [p, 1]$. *Ignore the indifference that occur on the boundaries of these sets.*

Part II

The Firm

10 The Firm's Problem

10.1 Technology

To study the problem of a firm, we first need to establish a framework to work in. A firm makes stuff. We need to model how this happens. The most general way to do this is through a **production possibilities set** $Y \subset \mathbb{R}^m$**.** Notice this is a subset of reals, including negatives. Here we use negatives to represent inputs and positives to represent outputs of production. For instance the vector $(-1,-1,1)$ says: take one input of goods 1 and 2 and create one output of good 3.

Example 73. *Two apples and one crust make a pie.* If x_1 is apples, x_2 is crusts, x_3 is pies, then the technology for producing pies can be represented by the set $Y = (-2,-1,1), (-4,-2,2), (-6,-3,3) ....$

If only one of the goods is ever an output and the rest are always inputs, we can represent production possibilities with a function. We call this the **production function.**

Example 74. *Two apples and one crust make a pie.* $f(x_1, x_2) = min\left\{\frac{1}{2}x_1, x_2\right\}$.

We will work with production functions rather than possibilities set through the rest of the class, but it is nice to be aware more flexible language exists.

10.2 Assumptions on $f(\boldsymbol{x})$

1. Continuous

2. Strictly Increasing

Our pie example is strictly but not strongly increasing.

This is essentially what we assumed about a utility function and the same reasons for making these assumptions generally apply. One major difference between production and utility is the ***production is inherently cardinal***. $f(x) = 2f(q)$ literally means that the input vector x produces twice the amount of output as q. Because of this, we cannot take monotonic transformations of production functions and have the represent the same technology. $2f\left(\right)$ is a technology that is twice as productive as $f\left(\right)$– not the same technology.

10.3 Cost Minimization / The Cost Function

The firm's problem is not as well defined as the consumer's problem. Generally, we assume firms maximize profits which is given by revenue minus cost. However, this maximization of profit will always involve minimizing the cost of producing whatever amount of output that is chosen at the profit maximizing level. Thus, cost minimization can be seen as a per-requisite for profit maximization. Letting w_i be the cost of input i, we can write the cost minimization problem this way:

Firm's Cost Minimization Problem: $Min_{x:f(x)\geq q}\left(\sum x_i w_i\right)$.

The x^* that solves this problem is called the **conditional factor demand** (analogous to the Hicksian demands for consumers). The value at the optimum is called the **cost function** (analogous to the consumer expenditure function):

Conditional Factor Demands: $x_i^*\left(q,w\right)$**.**
Cost Function: $c\left(q,w\right)$.

10.4 Properties of the Cost Function

The properties of the cost function are very similar to the consumer's expenditure function:

Properties of the cost function:
For $w \gg 0$

1. Continuous.
2. Strictly increasing in q.
3. Increasing in w.
4. Homogeneous of degree 1 in w.
5. Concave in w.
6. When x_i is single-valued, $\frac{\partial c(w,q)}{\partial w_i} = x_i\left(w,q\right)$.

10.5 An Example– Cobb Douglass Production.

Suppose the production function is $f\left(x\right) = x_1^\alpha x_2^\alpha$. The cost minimization Lagrangian is: $x_1 w_1 + x_2 w_2 - \lambda\left(x_1^\alpha x_2^\alpha - q\right)$. The derivatives are:

$$\frac{\partial\left(x_1 w_1 + x_2 w_2 - \lambda\left(x_1^\alpha x_2^\alpha - q\right)\right)}{\partial x_1} = w_1 - \alpha\lambda x_1^{\alpha-1} x_2^\alpha$$

$$\frac{\partial\left(x_1w_1+x_2w_2-\lambda\left(x_1^{\alpha}x_2^{\alpha}-q\right)\right)}{\partial x_2} \quad = \quad w_2-\alpha\lambda x_1^{\alpha}x_2^{\alpha-1}$$

Setting these equal to zero gives us the first order conditions: $\frac{w_1}{\alpha x_1^{\alpha-1}x_2^{\alpha}}=\lambda$, $\frac{w_2}{\alpha x_1^{\alpha}x_2^{\alpha-1}}=\lambda$. Notice that The $\frac{w_1}{\alpha x_1^{\alpha-1}x_2^{\alpha}}$ can be interpreted roughly as the cost of increasing output by one unit using input 1. $\frac{w_2}{\alpha x_1^{\alpha}x_2^{\alpha-1}}$ is the same for input 2. The first order condition says: *the cost of increasing output using any input is exactly the same and equal to* λ. λ is the "shadow cost" of production. The cost of increasing production at the optimum.

Solving the first order conditions gives $x_1=\frac{w_2}{w_1}x_2$ combining this with the production condition $x_1^{\alpha}x_2^{\alpha}=q$, gives us the the conditional factor demands $x_1=\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}q^{\frac{1}{2\alpha}}, x_2=\left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}q^{\frac{1}{2\alpha}}$. Notice that $x_i=x_i\left(1,w\right)q^{\frac{1}{2\alpha}}$. We will talk about this more in the next section.

The cost function can be derived by plugging these demands in to $w_1x_1+w_1x_2$. $c\left(q,w\right)=w_1\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}q^{\frac{1}{2\alpha}}+w_2\left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}q^{\frac{1}{2\alpha}}$. Simplifying this gives us $c\left(q,w\right)=2\left(w_1w_2\right)^{\frac{1}{2}}q^{\frac{1}{2\alpha}}$. Note that $2\left(w_1w_2\right)^{\frac{1}{2}}$ is $c\left(1,w\right)$ (the cost of producing one unit). Thus, $c\left(q,w\right)=c\left(1,w\right)q^{\frac{1}{2\alpha}}$.

10.6 Homogeneous/Homothetic Production

The demand and costs for a homogeneous production function is also homogeneous, but with a degree equal to the reciprocal of the production function. For instance, if production is homogeneous of degree $\frac{1}{2}$ then cost and demand are homogeneous of degree 2. This should be somewhat intuitive. Production is becoming less efficient according to a square-root function as output increases. Thus, the amount of input needed to increase production linearly will scale according to a the square of output.

Proposition 75. *For any* ***homogeneous*** *production function homogeneous of degree* β*...*

Factor demands are homogeneous of degree $\frac{1}{\beta}$ *in output:* $x_i\left(q,w\right)=q^{\frac{1}{\beta}}x_i\left(1,w\right)$.

Cost is homogeneous of degree $\frac{1}{\beta}$ *in output:* $c\left(q,w\right)=q^{\frac{1}{\beta}}c\left(1,w\right)$.

Exercise 76. Suppose the production function is $f\left(x\right)=\left(x_1^2+x_2^2\right)^{\frac{1}{3}}$. Solve for the conditional factor demands and cost functions and show they have this property.

Homothetic production has a similar, but slightly less useful property:

Proposition 77. *For any* ***homothetic*** *production function homogeneous of degree* β*:*

Factor demands can be separated into a function that depends only on q *and the demand for producing* 1 *unit:* $x_i=g\left(q\right)x_i\left(1,w\right)$.

Cost can be separated into a function that depends only on q *and the cost of producing* 1 *unit:* $c=g\left(q\right)c\left(1,w\right)$.

Exercise 78. Suppose the production function is $f\left(x\right)=ln\left(x_1\right)+ln\left(x_2\right)$. Solve for the conditional factor demands and cost functions and show they have this property.

10.7 Separability

Suppose the inputs of a production function can be partitioned and the production function written: $f(g_1(x_1, x_2), g_2(x_3, x_4))$. When this is possible, x_1, x_2 the relative amounts of x_1 and x_2 can be optimized without reference to x_3, x_4 and vise versa. That is, an optimal mix of the inputs within each group can be chosen independently. We will see an example of this below.

When a production function can be partitioned this way into groups the production function is **weakly separable** on those groups. In this case, it is weakly separable on x_1, x_2 and x_3, x_4. Another way to check for weak separability is to check that the ratio of partials of any two inputs in one group does not depend on the inputs in any other group. Note that:

$$\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = \frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}}, \frac{\frac{\partial f}{\partial x_3}}{\frac{\partial f}{\partial x_4}} = \frac{\frac{\partial g}{\partial x_3}}{\frac{\partial g}{\partial x_4}}$$

A production function is **strongly separable** if it is weakly separable into all possible groups of inputs. Alternatively, if the ratio of partials of any two inputs only depends on those inputs. For instance, take the production function $x_1^a x_2^b x_3^c x_4^d$. Note that $\frac{\frac{\partial\left(x_1^a x_2^b x_3^c x_4^d\right)}{\partial x_1}}{\frac{\partial\left(x_1^a x_2^b x_3^c x_4^d\right)}{\partial x_2}} = \frac{a x_2}{b x_1}$. These depend only on x_1 and x_2. A similar thing will hold for any pair.

10.8 A Separable Production Problem

When a production function is separable into groups, the groups can be optimized independently. Consider the function: $f(x_1, x_2, x_3, x_4) = ln(x_1) + ln(x_2) + \left(x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)$. This can be factored into separable groups x_1, x_2 and x_3, x_4. We can think of this problem as taking x_1, x_2 and producing intermediate input q_1 then taking x_3, x_4 and producing intermediate input q_2.

The production function for using these intermediate goods is then $f(q_1, q_2) = q_1 + q_2$. You can then use the cost function for producing intermediate goods q_1, q_2 to find the cost minimizing q_1 and q_2 and work backwards to get x_1, x_2, x_3, x_4.

Use this technique to solve the following problem. As a word of caution, there are some corner solutions involved.

Exercise 79. Suppose the production function is $f(x_1, x_2, x_3, x_4) = ln(x_1) + ln(x_2) + \left(x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)$. Solve this by using the technique of intermediate goods. Write down the conditional factor demands for x_1, x_2, x_3, x_4 as well as the cost function.

10.9 Exercises.

10.1. Suppose the production function is $f(x) = \left(x_1^2 + x_2^2\right)^{\frac{1}{3}}$. Solve for the conditional factor demands and cost functions and show they have the property given in *Proposition* 75.

10.2. Suppose the production function is $f(x) = \left(\frac{1}{x_1} + \frac{1}{x_2}\right)^{-1}$. Solve for the conditional factor demands and cost function and show they have the property given in *Proposition* 77.

10.3. Suppose the production function is $f(x) = x_1 + ln(x_2) + ln(x_3)$. Solve for the conditional factor demands and cost function.

10.4. Suppose the production function is $f(x_1, x_2, x_3, x_4) = (min\{x_1, x_2\})^{\frac{1}{2}} + x_3^{\frac{1}{3}} x_4^{\frac{1}{3}}$. Solve for the conditional factor demands and cost function when $w_1 = w_2 = w_3 = w_4 = 1$.

11 Profit

In the previous chapter we looked at cost minimization, which is a prerequisite for profit maximization. However, once this is done, the firm still needs to choose the optimal amount of output by solving: $\pi(q) = p(q)q - c(q)$. That is, revenue $p(q)q$ minus cost $c(q)$ where $c(q)$ is the cost function found in the last chapter. Notice that price is a function of the firm's output. However, we have not specified how to determine $p(q)$ just yet. We will do that in the next chapter. Generically, the firm solves:

Profit max problem: $Max_q p(q)q - c(q)$

For any interior solution, the first order condition is: $p'(q)q + p(q) = c'(q)$.

11.1 Perfect Competition

Generically, price is a function of a firms output. However, if every firm is so small that the market price will be approximately the same regardless of how much they produce, then we can have each firm assume the market price is fixed with respect to their output. We call this **perfect competition**. In that case, the profit function is $\pi(q) = pq - c(q)$ and the first order condition is $p = c'(q)$ (price equals marginal cost).

Example 80. Suppose a firm's production function is $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$ and is in perfect competition. Their cost function is $c(q, w) = 2(w_1 w_2)^{\frac{1}{2}} q^2$. The profit function is $\pi(q, w_1, w_2) = pq - 2(w_1 w_2)^{\frac{1}{2}} q^2$. The first order condition is $p = 4\sqrt{w_1 w_2} q$ and the optimal output is $q^* = \frac{p}{4\sqrt{w_1 w_2}}$. The firm's profit at the optimum is $\pi(q^*) = \frac{1}{8} \frac{p^2}{\sqrt{w_1 w_2}}$.

11.2 The Profit Function

The "profit function" is analogous to the cost function and the profit of the firm evaluated at the optimal q^* and cost minimized inputs. It is $\pi(p, w) = max_q pq - c(q) = Max_x\, pq - (w_1 x_1 + w_2 x_2)$. **When well defined** and under the assumption of **perfect competition** it has the following properties:

Properties of the profit function in perfect competition:

1. Increasing in p,
2. Decreasing in w,
3. Homogeneous of degree one in p, w
4. Convex in p, w (why)
5. Hotelling: $\frac{\partial \pi}{\partial p} = q\left(p, w\right), -\frac{\partial \pi}{\partial w_i} = x_i\left(p, w\right)$

Combining 4 and 5 we can prove that, output is increasing in price (weakly), and any input is weakly decreasing in it's own wage (this is the substitution effect for production) $\frac{\partial q}{\partial p} \geq 0$, $\frac{\partial x_i}{\partial w_i} \leq 0$.

Part III

Markets feat. Cournot

12 The Cournot Oligopoly Model

12.1 The Cournot Model

Suppose a number of firms each have cost function $c\left(q\right) = 2q^2$. If firms were price taker's each would assume assume profit is: $\pi = pq - 2q^2$. But consumer's are not willing to pay some fixed p for *any* market quantity. The quantity they will buy depends on the price and thus, the price they are willing to pay is a function of total quantity Q. This relationship is given by the inverse demand function. Let's assume it is $p = \frac{100}{Q}$. This is the most that firms could possibly charge and still sell total quantity Q.

Let's treat the inverse demand function as the relationship between price and market quantity and plug this into the profit function for each firm. Denote $\sum_{j \neq i} q_j = Q_{-i}$.[6] We can now write the profit function of a firm in **Cournot Oligopoly:**

Cournot Profit Function: $\pi_i\left(q_i, Q_{-i}\right) = p\left(q_i + Q_{-i}\right) q_i - c\left(q_i\right)$

12.2 An Example

Using the cost function and inverse demand from our working example, each firm has profit:

$$\pi_i\left(q_i, Q_{-i}\right) = \frac{100}{q_i + Q_{-i}} q_i - 2q_i^2$$

[6] For example if 10 firms all produce 10 units, $q_i = 10$ and $Q_{-i} = 90$ for all the firms.

Since this is concave, the first-order condition will give us the maximum for q_i. The first derivative is:

$$\frac{\partial\left(\frac{100}{q_i+Q_{-i}}q_i-2q_i^2\right)}{\partial q_i}=-\frac{100q_i}{(q_i+Q_{-i})^2}+\frac{100}{q_i+Q_{-i}}-4q_i$$

The first order condition is:

$$-\frac{100q_i}{(q_i+Q_{-i})^2}+\frac{100}{q_i+Q_{-i}}-4q_i=0$$

Notice this implicit function gives the optimal q_i as a function of Q_{-i}. But to solve this, we need to know how to deal with a situation where firm i's output decision q_i depends on the output decisions of all the other firms Q_{-i}. When the firms take prices as fixed, the quantity of other firms was irrelevant. However, now it is really important. Here, the payoffs of one firm, depend on the choices of the others. This is the realm of game theory. We won't be too formal about the game theory we do in this chapter, but we will get a feel for some of the fundamentals of game theory through these exercises.

12.3 Game Theory

Definition 81. A **game** is a set $\Gamma=\{P,S,\pi\}$. P is the set of players, S is the set of sets of strategies for each player, π are the payoffs functions mapping S into $\mathbb{R}^n$.

The scenario described above is a game. There are players (the firms), strategies (quantity choices for each firm), and payoffs (the profit functions of each firm). This game is called the **Cournot** model of oligopoly. Games are "solved" using **solution concepts**. A solution concept is a mapping from games into the set of strategies in those games. The most common solution concept in game theory is the **Nash Equilibrium**. Nash equilibrium refers to the best responses of players in a game. These are defined as follows:

Definition 82. The **best response** correspondence for player i: $B_i(s_{-i})$ is given by $B_i(s_{-i})=\{s_i|s_i\in S_i\, s_i\in arg.max_{S_i}\pi_i(s_i,s_{-i})\}$.

We can gather the best response correspondences of each player to get the the vector-valued best response correspondence:

Definition 83. The **best response** correspondence $B(s)$ is $B(s)=(B_1(s_{-1}),...,B_N(s_{-N}))$.

A Nash equilibrium requires that all players are best responding to each-others strategies.

Definition 84. A strategy profile $s=(s_1,...,s_n)$ is called a **Nash equilibrium** if $s\in B(s)$.

12.4 Nash Equilibrium in Cournot game.

In the case of our Cournot oligopoly game, Nash equilibrium requires be that all firms output decisions are optimal given what the other firms are doing. We have already seen the first order condition for the firms is: $\frac{100}{q_i+Q_{-i}} - \frac{100q_i}{(q_i+Q_{-i})^2} - 4q_i = 0$. This is the firm's (implicit) best response function. I will leave it implicit since the explicit solution has many terms.

Eh, what the heck... here it is...

$$q_i = \frac{1}{3}\left(-\frac{\sqrt[3]{2}Q_{-i}^2}{\sqrt[3]{-2Q_{-i}^3 - 675Q_{-i} + 15\sqrt{3}\sqrt{4Q_{-i}^4 + 675Q_{-i}^2}}} - 2Q_{-i} - \frac{\sqrt[3]{-2Q_{-i}^3 - 675Q_{-i} + 15\sqrt{3}\sqrt{4Q_{-i}^4 + 675Q_{-i}^2}}}{\sqrt[3]{2}}\right)$$

Assume there are two firms. The first order condition for firm 1 is given by the implicit function:

$$\frac{100}{q_1+q_2} - \frac{100q_1}{(q_1+q_2)^2} = 4q_1$$

This simplifies to:

$$q_1 = 25\frac{q_2}{(q_1+q_2)^2}$$

If we plug in a number for q_2 and solve for q_1, we will have 1's best response to 2 producing q_2. Suppose, for no reason at all that we assume $q_2 = 10$. Using a *Mathematica* to solve the first order condition for q_1, the best response is: $q_1 = 1.79652$. Is $(1.79652, 10)$ a Nash equilibrium? Let's see what 2's best response is to $q_1 = 1.79652$. Using mathematica again, it is $q_2 = 2.46875$. Thus, *this is not an equilibrium.*

What we need for an equilibrium is a set of (q_1, q_2) that solve the following implicit best response functions simultaneously:

$$q_1 = 25\frac{q_2}{(q_1+q_2)^2}, q_2 = 25\frac{q_1}{(q_1+q_2)^2}$$

Solving these for q_1 and q_2 we get only one solution: $(2.5, 2.5)$. **This is the Nash equilibrium of this game.**

12.5 Symmetric Equilibrium

Notice that this game has a **symmetric Nash equilibrium.** That is a Nash equilibrium where all the players choose the same strategy. We can find symmetric Nash equilibrium more easily by imposing symmetry on any one of the best response functions and setting $q_i = q$ for all i.

We can use symmetry to simplify the process of finding a Nash equilibrium for this game with J firms. To do this, we note that every best response function is:

$$q_i = 25\left(\frac{Q_{-i}}{(q_i + Q_{-i})^{\,2}}\right)$$

Imposing symmetry on this implicit best response function, we let $q_i = q$ for all i. This implies that $Q_{-i} = (J-1)\,q$. Thus, if there is a symmetric equlibrium, it has to solve:

$$q = 25\left(\frac{(J-1)\,q}{(Jq)^{\,2}}\right)$$

Solving this for q, we get:

$$q = \frac{5\sqrt{J-1}}{J}$$

At $J = 2$ for instance, $q = 2.5$.[7]

12.6 Comparison of Equilibrium in J

In the symmetric Cournot equilibrium we just found $q_{cournot} = \frac{5\sqrt{J-1}}{J}$. Let's compare for various J.

J	q_i	Q	p	π_i
2	2.5	5.	20.	37.5
5	2.	10.	10.	12.
10	1.5	15.	6.66667	5.5
100	0.497494	49.7494	2.01008	0.505
1000	0.158035	158.035	0.632772	0.05005

Notice how as J increases, the profit of each firm goes down and in the limit, each firm earns zero profit.

12.7 Fixed Point Theorems and Using Contraction to Find an Equilibrium

In general, a fixed point of a mapping $f : X \to X$ is an $x \in X$ such that $x \in f(x)$. Or, if f is not a set-valued mapping then $x = f(x)$.

Example 85. The fixed point of $f(x) = \frac{1}{2}\sqrt{x} + 1$ is $\frac{1}{8}\left(9 + \sqrt{17}\right) \approx 1.64039$

[7] To find the monopoly solution (J=1) we can reuse the above analysis. If we set $J = 1$ we are essentially just solving the profit maximizing level of output for a sole firm in a market. Notice when we set $J = 1$ in the above Cournot equilibrium we get that output is zero. What's going on? The issue is that demand is always unit-elastic. At any point where demand is either elastic or unit-elastic, a monopolist will have incentive to cut output by 1%, this will let them increase price by at least 1% which will keep revenue the same (or increase it) and lower cost. This will always increase profit. We need to be a bit more careful with market demand functions with monopoly.

Not all mappings have fixed points. $f(x) = x + 1$ has no fixed point. A **Nash Equilibrium** is a fixed point of the best response correspondence B. Strategy s is a **Nash Equilibrium** if $s \in B(s)$. How do we know that a particular game has a Nash equilibrium. Or, put another way, how do we know the best response correspondence associated with a game has a fixed point?

A *fixed point theorem* gives conditions under which a mapping has a fixed point and potentially gives some properties of that/those fixed points. Famously, John Nash applied the **Kakutani fixed-point theorem** to show that games have an equilibrium under some fairly general conditions. Still, not all games have an equilibrium. Even if they do have an equilibrium, they can be hard to find. I will leave it for your Game Theory course to look more deeply into the Kakutani fixed-point theorem and Nash's famous result. Instead, I'll use this space to talk about a different fixed point theorem.

Definition 86. A **Contraction** on a metric space (X, d) is a mapping $f : X \to X$ such that $\exists k \in [0, 1)$ such that $\forall x, x' \in X$, $d(f(x), f(x')) \leq kd(x, x')$.

A contraction reduces the distance between points in a very regular way. Because it keeps decreasing the distance between any two points, those two points have to keep getting closer to some point as we keep iterating the mapping. That point that we approach is a fixed point.

Exercise 87. Show that $f(x) = \frac{1}{2}\sqrt{x} + 1$ is a contraction on $[1, \infty)$.

Since $f(x) = \frac{1}{2}\sqrt{x} + 1$ is a contraction on $[1, \infty)$ it has a unique fixed point in that region that can be found by

Exercise 88. Use iteration to show that for $f(x) = \frac{1}{2}\sqrt{x} + 1$, the sequence $(2, f(2), f(f(2)), ...)$ is converging to $\frac{1}{8}\left(9 + \sqrt{17.0}\right) \approx 1.64039$.

Theorem 89. ***Banach fixed-point theorem.*** *Every contraction mapping on a non-empty complete metric space (X, d)has a unique fixed point $x^* \in X$ and $\forall x \in X$ the sequence $(x, f(x), f(f(x)), ...)$ converges to x^*.*

The bit about the iterative sequence is really the gem of the Banach fixed-point theorem because it tells us *how to find* the fixed point without actually solving for it analytically. This theorem is the key to using numerical tools (computer) to fix fixed points more easily.

Exercise 90. Use iteration to show that for $f(x) = \frac{1}{2}\sqrt{x} + 1$, the sequence $(2, f(2), f(f(2)), ...)$ is converging to $\frac{1}{8}\left(9 + \sqrt{17.0}\right) \approx 1.64039$.

12.8 Contraction and Cournot Equilibrium

Suppose we have the following Cournot model that differs from the running example in this chapter. Inverse market demand is $p(Q) = 100 - Q$ and cost of each firm is $c(q) = q^2$. If there are two firms, their best response functions are (note that the below best response should really stipulate that $q_i = 0$ when $25 - \frac{1}{2}q_j < 0$).

$$B\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right) = \begin{bmatrix} 25 - \frac{1}{4}q_2 \\ 25 - \frac{1}{4}q_1 \end{bmatrix}$$

Exercise 91. Show this function is a contraction on $X = [0,100] \times [0,100]$ using the definition of a contraction.

Showing that this "mapping" is a contraction using the definition above is not *that* difficult, but there is actually an easier way using the **Jacobian** of B which is the matrix of partial derivatives of B. In this case, the Jacobian is:

$$J = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$$

We have the following result:

Proposition 92. *For Convex $X \subset \mathbb{R}^n$ and function $f : X \to X$ that has continuous partial derivatives, let J be the Jacobian of f. If there exists a $k < 1$ such that $\left\|J\right\| \leq k$ for all $x \in X$ then f is a contraction.*

In the above proposition $\left\|J\right\|$ is some matrix norm of J. This could be the $\left\|J\right\|_1$ which is the maximum of the column sums of the absolute values of the entries, $\left\|J\right\|_\infty$ which is the maximum of the row sums of the absolute values of the entries, or $\left\|J\right\|_2$which is the square root of the sum of the squared terms. You could also choose a different norm if that is useful to the problem, but generally one of these three is easiest to show.

For this problem, we can really pick any of the norms. For instance, $\left\|J\right\|_1 = \left\|J\right\|_\infty = \frac{1}{4}$. Since these are both uniformly bounded below 1 everywhere, it is definitely true on $X = [0,100] \times [0,100]$. Thus, B is a contraction on $X = [0,100] \times [0,100]$.

Since we know that this best response function is a contraction, we can just get to iterating to find the equilibrium. Let's start at $(10,10)$.

$$B\left(\begin{bmatrix} 10 \\ 10 \end{bmatrix}\right) = \begin{bmatrix} 22.5 \\ 22.5 \end{bmatrix}, B\left(\begin{bmatrix} 22.5 \\ 22.5 \end{bmatrix}\right) = \begin{bmatrix} 19.375 \\ 19.375 \end{bmatrix}, B\left(\begin{bmatrix} 19.375 \\ 19.375 \end{bmatrix}\right) = \begin{bmatrix} 20.15625 \\ 20.15625 \end{bmatrix}$$

After only three iterations, we are already pretty close to the unique equilibirum of $(20,20)$.

As an aside, this also gives us a really easy way to show $f(x) = \frac{1}{2}\sqrt{x} + 1$ is a contraction on $[1,\infty)$. The Jacobian (which in this case is just the derivative of the function) is $\frac{1}{4x^{\frac{1}{2}}}$ which is at most $\frac{1}{4}$.

12.9 Exercises.

12.1. Show that $f(x) = \frac{1}{2}\sqrt{x} + 1$ is a contraction on $[1,\infty)$.

12.2. Use iteration to show that for $f(x) = \frac{1}{2}\sqrt{x} + 1$, the sequence $(2, f(2), f(f(2)), ...)$ is converging to $\frac{1}{8}\left(9 + \sqrt{17.0}\right) \approx 1.64039$.

12.3. Show the best response function $B\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right) = \begin{bmatrix} 25 - \frac{1}{4}q_2 \\ 25 - \frac{1}{4}q_1 \end{bmatrix}$ from the two firm cournot model presented above is a contraction on $X = [0,100] \times [0,100]$ using the definition of a contraction.

12.4. Suppose the inverse demand function in a market is $p(Q) = 50 - 2Q$. The cost function for any firm is $c(q) = 10 + 2q$.

A. Calculate the quantity chosen by a monopolist and the resulting market price.

B. Calculate the quantity chosen by each firm in an n firm Cournot equilibrium. What is the resulting market quantity and price as a function of n.

12.5 (Computer Required). Consider the Cournot model presented in this chapter with inverse demand $p(Q) = \frac{100}{Q}$ and firm cost function $c(q) = 2q^2$. If it costs 1 to enter and a firm will enter if it can earn positive profit and exit if will earn negative profit, how many firms will enter the market in equilibrium?

12.6. Two firms have production function $f(x_1, x_2) = \left(\frac{1}{x_1} + \frac{1}{x_2}\right)^{-1}$. Inverse demand is $p = \frac{400}{y_1 + y_2}$.

A. What are the conditional factor demands for x_1 and x_2.

B. What is the cost function for the firms?

C. Under what conditions would a price-taking firm choose to produce?

Now assume $w_1 = w_2 = 1$**.**

D. What are the firms' best response functions in a Cournot oligopoly model?

E. What are the Nash equilibria of this game?

12.7. Suppose there are two firms with different cost functions: $c_1(q_1) = 2q_1$, $c_2(q_2) = 3q_2$. What are the quantities of the two firms in Nash equilibrium a cournot game between them when inverse demand is $p(Q) = 100 - Q$?

13 Modified Cournot Models

13.1 Stackleberg Model of Sequential Quantity-Setting

In the standard Cournot model, firms move "simultaneously" that means, they choose their quantities not knowing the quantity of the other firms. The Stackleberg model relaxes, having the firms move in sequence. Here, we will look at an example with two firms.

Suppose cost is $2q^2$ with inverse demand of $p = 100 - Q$. Firm one moves first, choosing q_1. Firm 2 observes q_1 and then chooses q_2. What are q_1 and q_2 in equilibrium?[8]

The profit of both firms is $\pi_i = (100 - (q_i + q_j))\, q_i - 2q_i^2$. Let's work backwards to solve this game. Firm 2 observes q_1 and maximizes profit by solving the following first-order condition: $-q_1 - 6q_2 + 100 = 0$. The best response of firm 2 is: $q_2 = \frac{100 - q_1}{6}$.

Firm one knows firm two will use this best response so we can write firm one's profit using $\frac{100-q_1}{6}$ in place of q_2. We get firm one's profit purely in terms of q_1. $\pi_1(q_1) = \left(100 - \left(q_1 + \frac{100-q_1}{6}\right)\right) q_1 - 2q_1^2$. The first order condition is: $0 = \frac{1}{6}(q_1 - 100) - \frac{35q_1}{6} + 100$. The solution is $q_1 = 14.7059$. Plugging

[8]Technically we are solving for a **Subgame Perfect Nash equilibrium** in this sequential game. I will defer to letting you learn about the formal definition of this solution concept in Micro 2.

this back into firm 2's best response, we get $q_2 = 14.2157$. Notice, the first mover has an advantage here. Despite being identical, firm one earns more profit. $\pi_1 = 612.745, \pi_2 = 606.257$.

Exercise 93. Suppose there are three firms with cost functions $c(q) = 2q^2$ and inverse demand of $p(Q) = 100 - Q$. Firm one sets q_1. Firms 2 and 3 observe q_1 and then set q_2 and q_3. What is the subgame perfect Nash equilibrium?

13.2 Collusion and Repeated Games

Let's stick with demand of $100 - p$ and and cost $2q^2$ for each firm. Profit for each firm is given by: $\pi = (100 - Q_{-i} - q_i) q_i - 2q_i^2$. The first order condition is: $\frac{100-Q_{-i}}{6} = q_i$. Impose symmetry and solve to get the symmetric Nash equilibrium. This is the solution to the equation $\frac{100-(J-1)q}{6} = q$. Solving for q gives us the symmetric Nash solution of $\frac{100}{5+J} = q_{cournot}$.

What if instead of imposing symmetry on the first-order condition, we imposed symmetry before taking the first order condition? That is, imposed symmetry on the profit function to get $\pi = (100 - Jq) q - 2q^2$. Notice the profit of all firms is then $J\pi = J(100 - Jq) q - J2q^2$. Since this is just a scaling of the profit of a single firm, maximizing the individual firms profit by choosing q is also maximizing joint profits of *all the firms*. That is, by imposing symmetry before taking the first-order condition, we are essentially asking the firms to collude. Let's solve this for the collusive output: $\frac{\partial\left(100q-(J+2)q^2\right)}{\partial q} = 100 - 2(J+2)q$. $\frac{100}{2J+4} = q_{collusive}$.

If all of the firms are producing at $q_{collusive}$, is $q_{collusive}$ a best response? Definitely not since it is not a Nash equilibrium. The best response comes from the first order condition we found above: $\frac{100-Q_{-i}}{6} = q_i$. Plugging in the collusive output for all $J - 1$ other firms: $\frac{100-(J-1)\left(\frac{100}{2J+4}\right)}{6} = q_i$ simplifying this we get each firm's best response to the other's choosing the collusive output. $\frac{1}{6}\left(100 - \frac{100(J-1)}{2J+4}\right) = q_{best-response}$.

13.3 Collusion with 2 Firms.

Let's look at $J = 2$ using the analysis above. We get:

$$q_{cournot} \approx 14.2857$$
$$q_{collusion} = 12.5$$
$$q_{best-response} \approx 14.5833$$

Using this, we can calculate the profits of the firms under various conditions:

$$\pi_{cournot} = 612.25$$
$$\pi_{collusion} = 625$$

If a firm deviates from the collusive agreement by best-responding the profit of the two firms are:

$$\pi_{deviating} \approx 638.02$$

$$\pi_{non-deviating} \approx 598.96$$

In a one-shot version of this game, there is little hope a firm would stick to the collusive agreement. Those extra thirteen dollars are just too tempting. What would prevent a firm from deviating in the real world? Probably their loss of reputation. The other firm will never cooperate with them in the future. This story requires some ongoing relationship between the two firms. We can represent that in a model by having the firms play the Cournot game over and over again. A **infinitely repeated game**. Can collusion be enforced in equilibrium of infinitely repeated version of this game? It can, if the firms are patient enough.

Suppose each firm discounts future payoffs by β. For instance, the discount profit of colluding forever is:

$$(625) + \beta\,(625) + \beta^2\,(625) \ldots = \sum_{t=0}^{\infty} \beta^t\,(625) = \frac{625}{1-\beta}$$

A strategy in an infinitely repeated game is a choice of action given any history of play. On common type of agreement/strategy used in repeated games is called a "grim trigger" strategy. Here it takes the following form: each player agrees to collude forever as long no one has ever deviated. If someone has ever deviated they play the Cournot Nash equilibrium forever. This is a complete specification of what to do given any history of play, so it is a strategy. Given the strategies of this game, there are only two states the "game" can be in. 1) No one has deviated. 2) Someone has deviated. Given that players stick to the strategy, we can write down the profit of a firm that will result from playing according to the strategy in state S. The states are g for "good" where no one has deviated and b for "bad" where someone has deviated. For instance,

$$V\,(g) = \frac{625}{1-\beta}, V\,(b) = 612.25$$

With this we can write the profit U of a firm choosing action a (c for cooperate and d for deviate) in state s as follows

$$U\,(c, g) = 625 + \beta V\,(g)$$

$$U\,(d, g) = 638.02 + \beta V\,(b)$$

$$U\,(d, b) = 612.5 + \beta V\,(b)$$

We have now set this up as a so-called dynamic programming problem. While the problem of finding optimal strategies in infinite games seems quite difficult, using the *principle of optimal* in this dynamic programming interpretation simplifies the problem:

A strategy is optimal given the strategy of other player(s) if and only if there are no **one shot deviations**. A one-shot deviation is the change of a single action at one point in time, but subsequently following the prescribed strategy. Let's check for profitable one-shot deviations.

Let's see that the firms have incentive to stick with this agreement and continue colluding rather than deviating from the agreement. Sticking with the agreement and colluding forever will earn the firm 625 in every period. If the firm is going to do *anything else* in any period, it will trigger the Nash equilibrium. So if a firm is ever going to deviate, it might as well make the best of it and play the best response to the collusive quantity of the other firm *right now.* How much would it earn by doing this? If would get the best response profit this period of 638.02 and then get the Nash profit of 612.245 in every future period. This is $(638.021) + \sum_{t=1}^{\infty} \beta^t \, (612.245)$. So, for the firm to have inventive to not deviate, we need the following condition to be true:

$$625 + \sum_{t=1}^{\infty} \beta^t \, (625) > 638.021 + \sum_{t=1}^{\infty} \beta^t \, (612.245)$$

As long as $\beta > 0.50516$, the collusive agreement is enforced using the *grim trigger* strategy.

13.4 Another Example

Suppose the game is as follows:

	cooperate	defect
cooperate	10,10	0,20
defect	20,0	5,5

Grim Trigger

Let's check if Grim trigger can sustain cooperation. This will be true if:

$$10 + \beta \frac{10}{1-\beta} > 20 + \beta \frac{5}{1-\beta}$$

$$\frac{2}{3} < \beta < 1$$

Repentance.

Let's try another strategy I call "repentance".

Start in cooperative state. Cooperate unless last period was cooperative state and there was a deviation. If I deviated in that period, cooperate and then return to cooperative state. If I cooperated, deviate and then return to cooperative state. If we both deviated, cooperate and return to cooperative state.

There are technically a few states we need to check for one-shot deviations from, but let's start with being in a cooperative state. Cooperating in cooperative state and then going along with plan nets me: $10 + \beta \frac{10}{1-\beta}$. Deviating but then go along with plan requires that I cooperate tomorrow while opponent deviates, but then returning to cooperative state. This nets me: $20+\beta\left(0\right)+\beta^{2}\frac{10}{1-\beta}$. One-shot deviation is not beneficial if:

$$10+\beta\frac{10}{1-\beta}>20+\beta\left(0\right)+\beta^{2}\frac{10}{1-\beta}$$

$$10+\beta\frac{10}{1-\beta}>20+\beta^{2}\frac{10}{1-\beta}$$

$$\frac{10}{1-\beta}\left(\beta-\beta^{2}\right)>10$$

$$\frac{10}{1-\beta}\beta\left(1-\beta\right)>10$$

$$10\beta>10$$

Since $\beta<1$, this is never optimal! Thus, this is **not** an equilibrium strategy for any discount factor because the punishment is not harsh enough.

13.5 Exercises

13.1. Suppose there are two firms with cost functions $c\left(q\right)=2q^{2}$ and inverse demand of $p\left(Q\right)=100-Q$.

A. What is the Nash equilibrium of this cournot game?

B. What is the subgame perfect nash equilibrium if firm one moves first, firm two observes q_1 and then sets q_2?

13.2. Suppose two firms each earn 10 per day if they play the Cournot Nash equilibrium. If they collude, they can each earn 20 per day. But if one deviates and best-responds to this collusion, that firm will earn 40 while the other will earn only 5. Both firms have the same discount factor β. What does β need to be to sustain collusion?

14 Other Oligopoly Models

14.1 Bertrand Model

In the Cornot oligopoly model, firms commit to a quantity and let the market determine prices. What if they could commit to a price and let the market determine demand? Let's have a look.

Example 94. *Bertrand Oligopoly with 2 firms.*

There are 2 firms, strategies are a price for each firm $p_i \in \mathbb{R}_+$, each firm has constant marginal cost c and there are M consumers who will buy the good at the lowest price. Thus, payoffs are determined as follows. [9]

Payoffs are given as follows.

If $p_1 < p_2$, $\pi_1 = Mp_1 - Mc, \pi_2 = 0$.
If $p_2 < p_2$, $\pi_1 = 0, \pi_2 = Mp_1 - Mc$.
If $p_1 = p_2$, $\pi_1 = \frac{Mp_1 - Mc}{2}, \pi_2 = \frac{Mp_1 - Mc}{2}$.

Proposition 95. *The Nash equilibrium of this Bertrand game is $p_1 = p_2 = c$.*

Proof. Suppose $c < p_1 < p_2$. This is not an equilibrium since $\pi_2 = 0$ and firm 2 could earn strictly more by setting any price in the interval (c, p_1). Similarly, $c < p_2 < p_1$ is not an equilibrium. $p_1 < c$ or $p_2 < c$ cannot be an equilibrium since either of these conditions implies that some firm is earning negative profit and would earn zero by setting their price to $p_i = c$. $c < p_1 = p_2$ is also not an equilibrium. Either firm could earn more by setting a price in the interval $\left(\frac{p_i - c}{2}, p_i\right)$ since it will capture the entire demand and earn more profit. This leaves only one possibility $p_1 = p_2 = c$. Both firms earn zero profit. If either firm set a price above c, they will still get 0 profit. If any firm sets price below c, they will get negative profit. □

14.2 Hotelling Model

Suppose now that prices are fixed at 1 and cost is zero. The firms can locate themselves along a line $[0, 1]$. Consumers are uniformly distributed on this line and buy from the closest firm. Let l_1 and l_2 be the locations chosen by the firms. Without loss of generality, assume $l_1 \leq l_2$. The indifferent consumer is the consumer half-way between the two: $\tilde{x} = \frac{l_1 + l_2}{2}$. If $l_1 \neq l_2$ we get the profit functions: $\pi_1 = \left(\frac{l_1+l_2}{2}\right), \pi_2 = \left(1 - \frac{l_1+l_2}{2}\right)$We have to make a decision about what to do when $l_1 = l_2$. Let's just assume they split the market and get $\pi_1 = \frac{1}{2}, \pi_2 = \frac{1}{2}$.

What is the equilibrium of this game?

$l_1 \neq l_2$ cannot be part of an equilibrium.

Either firm has incentive to move towards the other and increase market share.

$l_1 = l_2 > \frac{1}{2}$ cannot be part of an equilibrium.

Either firm has incentive to move left a little.

$l_1 = l_2 < \frac{1}{2}$ cannot be part of an equilibrium.

Either firm has incentive to move right a little.

$l_1 = l_2 = \frac{1}{2}$ **is the only equilibrium.**

Holding $l_2 = \frac{1}{2}$ as fixed, firm 1's profit is $\pi_1 = \left(\frac{l_1 + \frac{1}{2}}{2}\right) = \frac{l_1}{2} + \frac{1}{4}$. Moving left decreases profit. Similarly hold $l_1 = \frac{1}{2}$ fixed. Increasing l_2 lowers π_2. Thus, $l_1 = \frac{1}{2}$ and $l_2 = \frac{1}{2}$ are mutual best responses and the only equilibrium of this game.

[9] We could complicate the model by having a more complex cost function and a demand function that is decreasing in the lowest price, but we will not do that here since it does not change the analysis much.

14.3 Combining These Models (*Linear Travel Cost*)

Two firms play a two-stage game. First, each picks a location on $[0, 1]$. Then, the firms choose their prices p_1, p_2. Consumers are uniformly distributed on $[0, 1]$ and every consumer minimizes price plus travel cost. In this case, assume travel cost is linear in distance to the firm $|x - l_i|$.

To solve this game, we start with the second stage pricing game. Suppose in stage one, the two firms are positioned at l_1 and l_2 with $l_1 < l_2$. Let's find the prices in Nash equilibrium. First, a useful lemma.

Lemma 96. *In equilibrium, p_1 and p_2 are such that $l_1 < \tilde{x} < l_2$.*

Proof. Suppose otherwise, then either $\tilde{x} > l_2$ or $\tilde{x} < l_1$. If $\tilde{x} > l_2$, then firm 2 earns $1 - \tilde{x} < 1 - l_2$ but if firm 2 chose $p_2 = p_1$ it would earn at least $1 - l_2$. Similarly if $\tilde{x} < l_1$, firm 1 earns $\tilde{x} < l_1$ but if firm 1 chose $p_1 = p_2$ then it would earn at least l_1. $\square$

Using this result, any equilibrium has $l_1 < \tilde{x} < l_2$. A consumer between l_1 and l_2 is indifferent between the two firms if $p_1 + (\tilde{x} - l_1) = p_2 + (l_2 - \tilde{x})$. Solving this for $\tilde{x}$ gives the identity of the indifferent consumer: $\tilde{x} = \frac{1}{2}(l_1 + l_2 - p_1 + p_2)$. Thus the profit of the firms are as follows:

$$\pi_1(p_1, p_2) = \left(\frac{1}{2}(l_1 + l_2 - p_1 + p_2)\right) p_1$$

$$\pi_2(p_1, p_2) = \left(1 - \frac{1}{2}(l_1 + l_2 - p_1 + p_2)\right) p_2$$

Taking first order conditions:

$$\frac{1}{2}(l_1 + l_2 - p_1 + p_2) - \frac{p_1}{2} == 0$$

$$\frac{1}{2}(-l_1 - l_2 + p_1 - p_2) - \frac{p_2}{2} + 1 == 0$$

Solving these for p_1 and p_2 gives us the Nash equlibrium prices.

$$p_1 = \frac{1}{3}(l_1 + l_2 + 2), p_2 = \frac{1}{3}(-l_1 - l_2 + 4)$$

Profits in equilibrium of the pricing game can then be written as follows after simplifying:

$$\pi_1 = \frac{1}{18}(l_1 + l_2 + 2)^2, \pi_2 = \frac{1}{18}(l_1 + l_2 - 4)^2$$

The firms know what will happen in the pricing game for any chosen locations. Thus, in the first stage, they choose locations to maximize the above profits.

However, here we find that firm 1 always has incentive to move towards firm 2 and firm 2 always has incentive to move towards firm 1. To see this note that $\frac{\partial \pi_1}{\partial l_1} > 0$ and $\frac{\partial \pi_2}{\partial l_2} < 0$. This led Hotelling in 1929 to conclude that the firms should locate at the same point. However, if they are positioned at the same point, their profits will be zero as in the Bertrand price-setting game. This leads to a situation in which there is no **pure strategy** equilibrium in this game. However, in 1979, d'Aspermont et al. showed that this issue does no arise when travel cost is quadratic.

14.4 Quadratic Travel Cost

Two firms play a two-stage game. First, each picks a location on $[0, 1]$. Then, the firms choose their prices p_1, p_2. Consumers are uniformly distributed on $[0, 1]$ and every consumer minimizes price plus travel cost. In this case, assume travel cost is linear in distance to the firm $(x - l_i)^2$.

To solve this game, we start with the second stage pricing game. Fix locations, the indifferent consumer is $\tilde{x} \to \frac{l_1^2 - l_2^2 + p_1 - p_2}{2(l_1 - l_2)}$. This leads to profit functions:

$$\pi_1 = p_1 \frac{l_1^2 - l_2^2 + p_1 - p_2}{2(l_1 - l_2)}$$

$$\pi_2 = p_2 \left(1 - \frac{l_1^2 - l_2^2 + p_1 - p_2}{2(l_1 - l_2)}\right)$$

These profits are concave in p_1 and p_2 respectively (this can be shown by taking the second derivative). Taking first order conditions and then solving the resulting system of best-responses gives the Nash equilibrium of the pricing game:

$$p_1 = \frac{1}{3}\left(-l_1^2 - 2l_1 + l_2^2 + 2l_2\right), p_2 = \frac{1}{3}\left(l_1^2 - 4l_1 - l_2^2 + 4l_2\right)$$

After simplifying, profits of the second stage game in equilibrium can be written as follows:

$$\pi_1(l_1, l_2) = \frac{1}{18}(l_2 - l_1)(l_1 + l_2 + 2)^2$$

$$\pi_2(l_1, l_2) = \frac{1}{18}(l_2 - l_1)(l_1 + l_2 - 4)^2$$

Unlike in linear travel cost, firm 1 has incentive to move away from firm 2 and firm 2 always has incentive to move away from firm 1. This leads to the equilibrium of $l_1 = 0, l_2 = 1, p_1 = 1, p_2 = 1$.

Appendix

Proofs

Proof. Proof of Proposition 7.

Since there exists some budget B that has an empty choice set, then for all x in B the upper contour set cannot be all of B. Otherwise x would be in the choice set and it would not be empty. Thus, $\forall x \in B, \#\left(\succsim(x)\right) < \#(B)$.

Thus, by completeness, for all x in B there is some other x' such that $x' \succ x$. That is, $\forall x \exists x' \in X : x' \succ x$. Choose an $x_1 \in B$, let $x_2 \in B$ be any element of $\succ(x_1)$. By the previous result, at least one must exist. We have $x_2 \succ x_1$. If there is an $x_3 \in B$ such that $x_3 \succ x_2$ and such that $x_1 \succ x_3$ we have identified a cycle. Otherwise, we continue with an inductive step. Suppose we have $x_n \succ ... \succ x_1$. $\succ(x_n)$ is non-empty. Either it contains an element x_{n+1} such that there is an $x_i \succ x_{n+1}$ in which case we have identified a cycle or it does not and we continue with another inductive step. Either we find a cycle or reach the N_{th} step with $x_N \succ x_{n-1} \succ ... \succ x_1$. $\succ(x_N)$ is non-empty. $\square$

Proof of Elasticity Relations

If instead we take the derivative of the budget constraint w.r.t. Income y:

To derive the first, take the derivative w.r.t. a price:

$$\frac{\partial q}{\partial p_i} = \frac{\partial \sum_{j \in I} p_j x_j(p, y)}{\partial p_i}$$

$$0 = \frac{\partial \sum_{j \in I} p_j x_j(p, y)}{\partial p_i}$$

$$0 = \sum_{j \neq i} p_j \frac{\partial x_j}{\partial p_i} + p_i \frac{\partial x_i}{\partial p_i} + x_i$$

$$0 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i} + x_i$$

$$-x_i = \sum_{i=1}^{n} p_j \frac{\partial x_j}{\partial p_i} \left(\frac{p_i}{x_j} \frac{x_j}{p_i}\right)$$

$$-x_i = \sum_{j=1}^{n} p_j \left(\frac{\partial x_j}{\partial p_i} \frac{p_i}{x_j}\right) \left(\frac{x_j}{p_i}\right)$$

$$-x_i = \sum_{j=1}^{n} p_j \left(\frac{x_j}{p_i}\right) \varepsilon_{j,i}$$

$$-x_i = \sum_{j=1}^{n} \left(\frac{p_j x_j}{p_i}\right) \varepsilon_{j,i}$$

$$-x_i = \frac{1}{p_i}\sum_{j=1}^{n}\left(\frac{p_j x_j}{1}\right)\varepsilon_{j,i}$$

$$-\left(\frac{1}{y}\right)\frac{x_i p_i}{1} = \left(\frac{1}{y}\right)\sum_{j=1}^{n}\left(\frac{p_j x_j}{1}\right)\varepsilon_{j,i}$$

Denote $\frac{x_i p_i}{y} = s_i$ (the share of income spend on i):

$$-s_i = \sum_{j=1}^{n}\left(\frac{p_j x_j}{y}\right)\varepsilon_{j,i}$$

$$y = \sum_{j\in I} p_j x_j\left(p, y\right)$$

$$\frac{\partial y}{\partial y} = \frac{\partial \sum_{j\in I} p_j x_j\left(p, y\right)}{\partial y}$$

$$1 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial y}$$

$$1 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial y}\left(\frac{y}{x_j}\frac{x_j}{y}\right)$$

$$1 = \sum_{j=1}^{n}\frac{p_j x_j}{y}\eta_j$$

Monotonic Functions

For these definitions, first we need to define some relations on vectors. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$.

$x \geq y \Leftrightarrow x_i \geq y_i, \forall i \in \{1, ..., n\}$

$x > y \Leftrightarrow x_i \geq y_i, \forall i \in \{1, ..., n\}$ and $\exists j \in \{1, ..., n\}$ such that $x_j > y_j$

$x \gg y \Leftrightarrow x_i > y_i, \forall i \in \{1, ..., n\}$

Definition 97. Increasing Function. A function $f\left(x_1, ..., x_n\right)$ is increasing if and only if $f\left(x\right) \geq f\left(y\right)$ for all $x \geq y$

We can strengthen increasing to *strictly increasing as follows:*

Definition 98. Strictly Increasing Function. A function $f(x_1, ..., x_n)$ is strictly increasing if and only if it is increasing and $f(x) > f(y)$ for all $x \gg y$

We can strengthen increasing to *strongly increasing as follows:*

Definition 99. Strongly Increasing Function. A function $f(x_1, ..., x_n)$ is strongly increasing if and only if it is increasing and $f(x) > f(y)$ for all $x > y$

$$x > y$$

Convex Sets, Convex/Concave Functions, Quasi-Convex/Concave Functions

In a subset of euclidean space X, the line between $x \in X$ and $x' \in X$ is another point in the set X given by $tx + (1-t)x'$ where $t \in [0,1]$. We call points like this **Convex Combinations** of x and x'.

Example 100. Convex Combination. Let $x = (1,0), x' = (0,1)$. The convex combinations of x, x' are $t(1,0) + (1-t)(0,1)$ at $t = 0.5$ on this line, we have the point $(0.5, 0.5)$.

Definition 101. Convex Set. $S \subseteq X$ is convex if it contains all of its convex combinations: $\forall x, x' \in S$, $\forall t \in [0,1]$, $tx + (1-t)x' \in S$.

There is also the concept of convexity on functions:

Definition 102. Convex Function. $f(x)$ is **Convex** if $\forall x, x' \in X, t \in [0,1], tf(x) + (1-t)f(x') \geq f(tx + (1-t)x')$ it is **Strictly Convex** if the inequality is strict for $t \in (0,1)$.

Fact 103. A convex function has convex lower contour sets.

Definition 104. Concave Function. $f(x)$ is **Concave** if $\forall x, x' \in X, t \in [0,1], tf(x) + (1-t)f(x') \leq f(tx + (1-t)x')$ it is **Strictly Concave** if the inequality is strict for $t \in (0,1)$.

Fact 105. A concave function has convex upper contour sets.

The contour set facts above are only implied by convex and concave functions respectively, but those are important enough properties that we define the class of all functions with these properties.

Definition 106. Quasi-Concave Function. $f(x)$ is quasi-concave **if and only** it has convex upper contour sets.

Fact 107. A function $f(x)$ is **quasi-concave** if and only if is a monotonic transformation of a concave function. A function $f(x)$ is **quasi-concave** if and only if $f(tx + (1-t)x') \geq min\{f(x), f(x')\}$ for $t \in [0,1]$. It is **strictly quasi-concave** if and only if the inequality is strict for $t \in (0,1)$.

Quasi-convexity follows in an analogous way from relaxing convex functions.

Homogeneous/Homothetic Functions

Homogeneous Functions:

$$f(t\boldsymbol{x}) = t^{\alpha} f(x)$$

f is homogeneous of degree α.

Math Notation

Sets and Elements

X: Set

x: Element of a Set

$\mathcal{X}$: Set of Sets

$X = \{x|...\}$: Set-Builder Notation

Some Special Sets

$\mathbb{R}$: The Real Numbers

$\mathbb{N}$: The Natural Numbers

$\mathbb{R}_{+}$: The Non-negative Real Numbers

$\mathbb{R}_{++}$: The Strictly Positive Real Numbers

Set Operators

$\subseteq$: Weak Subset

$\subset$: Strict Subset

$\cap$: Set Intersection

$\cup$: Set Union

$P(X)$: Power Set of X

$\#(X)$: Cardinality of X

First-Order Logic

$\wedge$: Logical And

$\vee$: Logical Or

$\exists$: There Exists

$\forall$: For All