1 Homogeneous & Homothetic Functions

Theorem 1. *Homogeneity of the partial derivative of a homogeneous function*

If f: $\mathbb{R}^n_{++} \rightarrow \mathbb{R}$ is continuously differentiable and homogeneous of degree α, then each partial derivative f_i is homogenous of degree $\alpha - 1$.

Theorem 2. *Euler's Principle*

$f(x)$ is homogeneous of degree k iff $\forall$ x

$$\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i = kf(x)$$

Facts 1. *About Homogeneous Functions*

- A positive, quasi-concave function that is homogeneous of degree $k \in (0, 1]$, is concave.
- A positive, strictly quasi-concave function that is homogeneous of degree $k \in (0, 1)$, is strictly concave.
- If a demand function is homogeneous in income, then the income elasticity of demand is equal to 1.
- If a production function, $q = f(x)$ is HD k, then the corresponding cost function, $c(w, y)$, is HD $\frac{1}{k}$ in production.
- Linear homogeneous functions will always be concave (proved by Shephard).

Definition 1. *Homothetic Functions*

A function is homothetic if

- A monotonic transformation of a linearly homogenous function

Facts 2. *About Homothetic Functions*

- A monotonic transformation of a homothetic function is homothetic.
- If a function is homothetic, then $f(x) = f(y) \Rightarrow f(\lambda x) = f(\lambda y)$, for all $\lambda > 0$.
- If it is homothetic and cont. differentiable, then for any $x \in \mathbb{R}_{++}$ and $\lambda > 0$, $\exists$ a $k > 0$, such that $\nabla f(x) = k\nabla f(\lambda x)$.
- Hence, if a utility function is continuously differentiable and homothetic, then for any $\lambda > 0$, $MRS_{ij}(x) = MRS_{ij}(\lambda x)$.
- A homogeneous function of some degree $k \neq 0$ is homothetic.
- CES functions are homothetic

2 Concavity & Convexity

Definition 2. *Concave Functions*

1. A real valued function defined on a convex set $A \subset \mathbb{R}^n$ is concave, if

$$f(tx^{'} + (1-t)x) \geq tf(x^{'}) + (1-t)f(x) \tag{1}$$

   for all $t \in [0,1]$. The function is strictly concave if the inequality is strict for $t \in (0,1)$.

2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is concave iff its Hessian, $D^2 f(x)$, is negative semi- definite. A function is strictly concave if its Hessian is negative definite.

Definition 3. *Quasi-Concave Functions*

1. A function defined over a convex set is quasi-concave, if its upper contour sets are convex.
2. A function is quasi-concave iff $f(x^\alpha) \geq Min\{f(x), f(x')\}$.
3. A function is quasi-concave iff $f(x) \geq f(x')$ implies $f(x^\alpha) \geq f(x')$ for $\alpha \in [0,1]$.
4. A function is quasi-concave iff its bordered Hessian is negative semi-definite.
5. A function is quasi-concave iff every monotonic transformation of it is quasi-concave.

Facts 3. *About Concave & Quasi-Concave Functions*

- Any monotonic transformation of a concave function is quasi-concave.
- Any monotonic transformation of a strictly concave function is strictly quasi-concave.
- If $f(x) = (\sum \alpha_i x_i^\rho)^\beta$. If $\rho = 1$, f is quasi-concave. If $\rho < 1$, f is stricly quasi- concave. If $\rho \leq 1$ and $\beta\rho \leq 1$, then f is concave. If $\rho < 1$ and $\beta\rho < 1$, f is strictly concave.
- A linear combination of concave functions is concave.
- Every Cobb-Douglas function with positive parameters is quasi-concave
- The level sets of a quasi-concave function are convex. (i.e. the indifference curves are convex to the origin).
- If a function is quasi-concave, the solution for the local maximum, must also be a global maximum.

Definition 4. *Convex Functions*

1. A real valued function defined on a convex set $A \subset \mathbb{R}^n$ is convex, if

$$f(tx^{'} + (1-t)x) \leq tf(x^{'}) + (1-t)f(x) \tag{2}$$

   for all $t \in [0,1]$. The function is strictly convex if the inequality is strict for $t \in (0,1)$.

2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex iff its Hessian, $D^2 f(x)$, is positive semi- definite. The function is strictly quasi-convex if its Hessian is positive definite.

Definition 5. *Quasi-Convex Functions*

1. A function defined over a convex set is quasi-convex, if its lower contour sets are convex.
2. A function is quasi-convex iff $f(x^\alpha) \leq Max\{f(x), f(x')\}$.
3. A function is quasi-convex iff $f(x) \geq f(x')$ implies $f(x^\alpha) \leq f(x)$ for $\alpha \in [0,1]$.

4. A function is quasi-convex iff its bordered Hessian is positive semi-definite.

Facts 4. *About Convex & Quasi-Convex Functions*

See 'Facts about Concave & Quasi-Concave Functions' above, the results are similar.

3 Linear Algebra - Definiteness of Matricies

Definition 6. *Postive & Negative (Semi) Definiteness*

Positive (Semi) Definite Matricies

1. A symmetric n x n matrix, A is positive semi-definite iff for all $z \in \mathbb{R}^n \neq 0$, $z^T Az \geq 0$. The matrix is positive definite if the inequality is strict.
2. A n x n matrix, A, is positive semi-definite iff all its n leading principle minors are positive. The matrix is positive definite if all its n leading principle minors are strictly positive.

Negative (Semi) Definite Matricies

1. A symmetric n x n matrix, A is negative semi-definite iff for all $z \in \mathbb{R}^n \neq 0$, $z^T Az \leq 0$. The matrix is negative definite if the inequality is strict.
2. A n x n matrix, A, is negative semi-definite iff its odd leading principle minors are negative and its even leading principle minors are positive in sign. The matrix is negative definite if all its these signs hold strictly.

This test comes in handy in determining the concavity and convexity of a multivariate function, the definiteness check of the Slutsky substitution matrix, or a myriad of other useful theorems.

Example Let A be an 2 x 2 matrix, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The definiteness check, $z^T Az$ can be obtained by solving

$$\begin{pmatrix} z_1 & z_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

which yields $z_1^2 a_{11} + z_1 z_2 a_{21} + z_1 z_2 a_{12} + z_2^2 a_{22}$. So the rest is to check if this is greater than, less than, or equal to zero.

4 Elasticity and CES functions

Definition 7. *Elasticity of a function of a single variable*

$$\eta(x) = \frac{xf'(x)}{f(x)} = \frac{df(x)/f(x)}{dx/x} = \frac{dlnf(x)}{dlnx}$$

This measures the responsiveness of $f(x)$ from a percent change in x. This is beneficial because this measure is unit invariant (unlike the simple derivative). It is important to notice that this value is dependent on x, or more generally on its arguments.

Facts 5. *About Elasticity*

- If a function, $f(x) = y$ is strictly monotonic, then it has a well defined inverse such that, $\phi(y) = x$, whenever $f(x) = y$. If $\eta(x)$ is the elasticity of f, then $\frac{1}{\eta(x)}$ is the elasticity of $\phi(y)$ w.r.t y.
- Applicable to monopoly theory. Does a monopolist's price revenue increase or decrease if price increases?

$$\frac{dlnR(p)}{dlnp} = 1 + \frac{dlnD(p)}{dlnp} = 1 + \eta(p)$$

Therefore profit will increase iff $\eta(p) > -1$ and will decrease if $\eta(p) < -1$.

Definition 8. *Elasticity of Substitution*

The elasticity of substitution between two factor inputs, measures the percentage response of the relative marginal products of the two factors to a percentage change in the ratio of their quantities. We require the production function to be continuously differentiable and strictly quasi-concave.

$$\sigma_{ij} = \frac{dln(x_j/x_i)}{dln(f_i(x)/f_j(x))} = \frac{d(x_j/x_i)}{x_j/x_i}\frac{f_i(x)/f_j(x)}{d(f_i(x)/f_j(x))}$$

When $\sigma = 0$, the goods are perfect complements and when $\sigma \to \infty$, the goods are perfect substitutes. If the function is quasi-concave, $\sigma \geq 0$.

Lemma 1. *Constant Elasticity of Substitution*

A function $f : \mathbb{R}^n \to \mathbb{R}$ has constant elasticity of substitution if it is a monotonic transformation of one of these two forms

$$f(x_1, \ldots, x_n) = A\left(\sum_{i=1}^{n} \lambda_i x_i^{\rho}\right)^{\frac{k}{p}}$$

$$f(x_1, \ldots, x_n) = A\prod_{i=1}^{n} x_i^{\lambda_i}$$

where $A > 0, k > 0$, and λ is non-negative for all i and add to one. Linear Cobb-Douglas is the case where $\rho \to 0$. Hence, the degree of substitutability between factors is always constant, regardless of the level of input proportion.

Facts 6. *About CES Functions*

- CES functions must be monotonic transformations of homogeneous functions, $k \neq 0$.
- CES functions have the homothetic property.
- Not all homogeneous functions are of the CES form
- A monotonic transformation of a CES function has CES properties.
- If a production function is of the CES form, then its elasticity of substitution is $\sigma = \frac{1}{1-\rho}$.
- If a production function is CES with elasticity of sub. equal to σ, then the cost function will be CES with elasticity of substitution equal to $\frac{1}{\sigma}$.

$$c(p) = A\left(\sum_{i=1}^{n} \lambda_i p_i^{\rho}\right)^{\frac{1}{r}}$$

with $r = \frac{\rho}{\rho-1}$.

5 Separability

Definition 9. *Separable preferences*

Preferences, R, are separable on M if whenever is it true that

$$(x_M, x_{\sim M})R(x'_M, x_{\sim M})$$

it must be true that

$$(x_M, x'_{\sim M})R(x'_M, x'_{\sim M})$$

Theorem 3. *Test for Separable Preferences*

If there are n commodities and preferences are represented by $u(x)$, then preferences are separable on M iff $\exists$ a real-valued aggregator function, $f(x_M)$ such that $u(x_1, \ldots, x_n) = U(f(x_M), x_{m+1}, \ldots, x_n)$ where U must be increasing in f.

<u>Note:</u> Hence, if we can separate a subset of goods into a function, and utility is increasing in that function, then preferences are separable on that particular subset. If we can group every good into its own function, then preferences would be separable in all goods. Remember, separability doesn't have to mean that these functions are *additively separable*.

Definition 10. *Strongly Separable Functions*

A function, $f(x) : \mathbb{R}^n \to \mathbb{R}$ is strongly separable if there exists a partition $N_1, \ldots, N_s$ of the arguments, with $s > 1$, such that the ratio of derivatives $\frac{f_i(x)}{f_j(x)}$ for any two goods in **separate** subgroups, is not dependent on the quantity goods from any other subgroup.

Definition 11. *Weakly Separable Functions*

A function, $f(x) : \mathbb{R}^n \to \mathbb{R}$ is weakly separable if there exists a partition $N_1, \ldots, N_s$ of the arguments, with $s > 1$, such that the ratio of derivatives $\frac{f_i(x)}{f_j(x)}$ for any two goods in the **same** subgroup, is not dependent on the quantity goods from any other subgroup.

Definition 12. *Additively Separable Functions*

A function $f : \mathbb{R}^n \to \mathbb{R}$ is additively separable if it can be expressed as

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} g_i(x_i)$$

Additively separable preferences follow this form as well. We only require that each $g_i(\cdot)$ be real valued and differentiable.

Facts 7. *Easy Check for Additively Separable Preferences*

- Preferences are additively separable iff every subset of commodities represented by a utility function is separable.

This means that we can take any collection of commodities in the utility function, and that set is separable compared to the leftover set.

Theorem 4. *Affine Transformations of Additively Separable Functions*

If two additively separable utility functions, $U = \sum_i u_i(x_i)$ and $V = \sum_i v_i(x_i)$, represent the same preferences, then one must be an affine transformation of the other, or

$$U = aV + b$$

or this can be arranged as

$$V = \frac{1}{a}U - \frac{b}{a}$$

Both are affine transformations of each other. Why is this useful? If a utility function is additively separable, then any affine transformation of that function will also represent the consumer's preferences. Therefore we can only talk about ordinal preferences (i.e. utility has no real meaning other than comparing two bundles for a single consumer).

Theorem 5. *Homothetic and Additively Separable Preferences*

If preferences are representable by a utility function, then they are homothetic and additively separable iff they take one of these two forms

$$u(x) = \sum a_i x_i^b$$

$$u(x) = \sum a_i ln(x_i)$$

Notice that if a utility function takes the CES form, $u(x) = \left(\sum a_i x_i^b\right)^{\frac{1}{b}}$, then a monotone transformation of that function (i.e. taking it two the power b) will yield an additively separable function. Hence,

- CES functions are additively separable up to a monotonic transformation

6 Utility Maximization & Cost Minimization

Facts 8. *About Marshallian Demand,* $x(p, y)$

- HDZ in (p, y)
- If $u(x)$ is quasiconcave, $x(p, y)$ is a convex set.

Facts 9. *About Indirect Utility,* $v(p, y)$

- Continuous on $\mathbb{R}_{++}$ x $\mathbb{R}_{+}$
- Homogeneous of degree zero in(p, y)
- Strictly increasing in y
- Decreasing in p
- Roy's Identity

$$x_i(p, y) = -\frac{\partial v(p, y)/\partial p_i}{\partial v(p, y)/\partial y}$$

Facts 10. *About Hicksian Demand,* $x^h(p, u)$

- HDZ in prices, p
- If u(x) is quasiconcave, $x^h(p, y)$ is a convex set.

Facts 11. *About the Expenditure Function,* $e(p, u)$

- Homogeneous of degree 1 in p.
- Increasing in p
- Concave in p
- Shephard's Lemma

$$\frac{\partial e(p, u)}{\partial p_i} = x_i^h(p, u)$$

Definition 13. *Indirect Utility of the Gorman Form*

Indirect utility has the Gorman polar form if it takes the form of

$$v_i(p, y_i) = a_i(p) + b(p)y_i$$

Notice that if all individuals have Gorman indirect utility functions, that utilitarian aggregation, $V(p, y) = \sum_i v_i(p, y_i)$, where $y = \sum_i y_i$, yields an aggregate indirect utility function of the same form

$$\sum_{i=1}^{n} v_i(p, y_i) = \sum_i \big(a_i(p) + b(p)y_i\big) = \sum_i a_i(p) + b(p)\sum_i y_i = A_i(p) + b(p)y$$

where $A_i(p) = \sum_i a_i(p)$. Notice that if we have indirect utility of the Gorman polar form, then we can evaluate welfare using a normative representative consumer. Also, the exact distribution of wealth does not matter, only aggregate wealth (since our aggregate indirect utility function is only dependent on y not on the y_i's.

Facts 12. *Duality*

1. $x_i^h(p,u) = x_i(p, e(p,u))$
2. $x_i(p,y) = x_i^h(p, v(p,y))$
3. $v(p, e(p,u)) = u$
4. $e(p, v(p,y)) = y$

Definition 14. *Slutsky Decomposition*

For Marshallian demand $x(p,y)$, we have the Slutsky decomposition of

$$\frac{\partial x_i(p,y)}{\partial p_j} = \frac{\partial x_i^h(p,u)}{\partial p_j} - x_j(p,y)\frac{\partial x_i(p,y)}{\partial y}$$

The proof is accomplished by taking the duality fact that $x_i^h(p,u) = x_i(p, e(p,u))$, then take the derivative w.r.t p_j. The rest of the proof relies on Shephard's Lemma.

Definition 15. *Cournot and Engel aggregation*

Let η_i, ϵ_{ij}, and s_i, be the income elasticity, price elasticity, and income share respectively.

Cournot Aggregation: $\sum_i s_i \epsilon_{ij} = -s_j$

Engel Aggregation: $\sum_i s_i \eta_i = 1.$

Both can be derived from taking derivatives of $y = \sum_i p_i x_i(p,y)$ with respect to p_j and y respectively. Cournot says that total expenditure cannot change in response to a change in prices. Engel says that total expenditure must change in exact proportion to any wealth change.

7 Production Theory

Facts 13. *About the Cost Function,* $c(w, y)$

- Continuous over the domain
- Strictly increasing and unbounded above in y for all positive factor prices
- Increasing in w
- HD 1 in w
- Concave in w
- Shephard's Lemma

Facts 14. *About the Conditional Input Demand,* $x(w, y)$

- HDZ in input prices w
- The substitution matrix composed of $\frac{\partial x_i(w,y)}{\partial w_j}$ is symmetric and negative semi-definite

Scale Properties

Definition 16. *Global Returns to Scale*

A production functions $f(x)$ has global returns to scale that are

1. Constatnt if $f(tx) = tf(x)$
2. Increasing if $f(tx) > tf(x)$
3. Decreasing if $f(tx) < tf(x)$

Note: Notice that homogeneity properties will give us information about the global scale properties, but we cannot guarantee the homogeneity property of a any scale function. In other words we can go from (homogeneity $\Rightarrow$ scale properties), but not (scale properties $\Rightarrow$ homogeneity).

Definition 17. *Local Returns to Scale*

A production function, $f(x)$, has constant, increasing, or decreasing local returns to scale around a point x, if $\mu(x)$ is equal to, greater than, or less than 1. Where

$$\mu(x) = \frac{\sum_{i=1}^{n} f_i(x)x_i}{f(x)}$$

Notice that a function's local homogeneity will tell you about its local scale properties.

8 Expected Utility

Definition 18. *Risk Behavior*

A consumer with expected utility defined over wealth has risk behavior according to

1. Risk Averse at $w \quad \Rightarrow \quad u(E(w)) > u(g)$
2. Risk Neutral at $w \quad \Rightarrow \quad u(E(w)) = u(g)$
3. Risk Loving at $w \quad \Rightarrow \quad u(E(w)) < u(g)$

Hence, we can use Jensen's inquality to determine the local risk behavior of a consumer.

Definition 19. *Arrow-Pratt Measure of Absolute and Relative Risk Aversion*

Measure of Absolute Risk Aversion

$$R_a(w) = -\frac{u''(w)}{u'(w)}$$

Measure of Relative Risk Aversion

$$R(w) = -\frac{u''(w)w}{u'(w)}$$

These measures can determine if an expected utility maximizer has CARA, DARA, or IARA (and likewise CRRA, DRRA, IRRA).

Definition 20. *Certainty Equivalent and Risk Premium*

The certainty equivalent of a simple gamble, g, over wealth levels satisfies

$$u(CE) = u(g)$$

It has the interpretation that the consumer would be indifferent between taking the certainty equivalent to taking a certain simple gamble, g.

The risk premium for any simple gamble, g, satisfies

$$u(g) = u(E(g) - RP)$$
$$or$$
$$RP = E(g) - CE$$

Therefore, we can see that the sign of the risk premium will change depending the the risk behavior of the consumer.