1 More Complex Optimization Examples

1.1 Some Examples with Multiple Constraints

Maximize x_1x_2 subject to (1) $\left(x_1^2+x_2^2\right)^{\frac{1}{2}} \leq 10$ and (2) $2x_1+x_2 \leq 40$.

$$x_1x_2-\lambda_1\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)-\lambda_2\left(2x_1+x_2-40\right)$$

$$\frac{\partial\left(x_1x_2-\lambda_1\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)-\lambda_2\left(2x_1+x_2-40\right)\right)}{\partial x_1} = -\frac{\lambda_1x_1}{\sqrt{x_1^2+x_2^2}}-2\lambda_2+x_2$$

$$\frac{\partial\left(x_1x_2-\lambda_1\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)-\lambda_2\left(2x_1+x_2-40\right)\right)}{\partial x_2} = -\frac{\lambda_1x_2}{\sqrt{x_1^2+x_2^2}}-\lambda_2+x_1$$

$$-\frac{\lambda_1x_1}{\sqrt{x_1^2+x_2^2}}-2\lambda_2+x_2=0$$

$$-\frac{\lambda_1x_2}{\sqrt{x_1^2+x_2^2}}-\lambda_2+x_1=0$$

$$\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10=0$$

Notice, we can't be on the boundary of both. It is infeasible.
Let's try checking the boundary of constraint 2 and interior of 1.

$$-\frac{\lambda_1x_1}{\sqrt{x_1^2+x_2^2}}-2\lambda_2+x_2=0$$

$$-\frac{\lambda_1x_2}{\sqrt{x_1^2+x_2^2}}-\lambda_2+x_1=0$$

$$\{\{x_1\to 10, x_2\to 20, \lambda_2\to 10\}\}$$

$$(10, 20)$$

$$22.3607$$

This violates constraint 1 and is thus infeasible.
On the boundary of constraint 1 and interior of 2.

$$x_1 \to 7.07107, x_2 \to 7.07107, \lambda_1 \to 10$$

$$21.2132$$

$$Solve[\{-\frac{\lambda_1 * x_2}{\sqrt{x_1^2+x_2^2}} + x_1 == 0,, \left(x_1^2+x_2^2\right)^{\frac{1}{2}} - 10 == 0\}, \{x_1, x_2, \lambda_1\}]$$

Maximize $x_1 x_2$ subject to (1) $\left(x_1^2+x_2^2\right)^{\frac{1}{2}} \leq 10$, (2) $2x_1 + x_2 \leq 15$.

$$-\frac{\lambda_1 x_1}{\sqrt{x_1^2+x_2^2}} - 2\lambda_2 + x_2 = 0$$

$$-\frac{\lambda_1 x_2}{\sqrt{x_1^2+x_2^2}} - \lambda_2 + x_1 = 0$$

$$\left(x_1^2+x_2^2\right)^{\frac{1}{2}} - 10 = 0$$

$$2x_1 + x_2 - 15 = 0$$

Maximize $x_1 x_2$ subject to (1) $\left(x_1^2+x_2^2\right)^{\frac{1}{2}} \leq 10$, (2) $2x_1 + x_2 \leq 20$.

$\{\{x_1 \to 6., x_2 \to 8., \lambda_1 \to 4., \lambda_2 \to 2.8\}, \{x_1 \to 10., x_2 \to 0., \lambda_1 \to -20., \lambda_2 \to 10.\}\}$

$$x_1 \to 6., x_2 \to 8., \lambda_1 \to 4., \lambda_2 \to 2.8$$

1.2 Formally Dealing with Non-Negativity Constraints

Maximize $u = \log(x_1) + \sqrt{x_2} + x_3$ subject to $x_1 + x_2 + x_3 \leq m$ and $x_1, x_2, x_3 \geq 0$