1 More Complex Optimization Examples

1.1 Formally Dealing with Non-Negativity Constraints

1.2 Quasi-Linear

Maximize $u = \log(x_1) + x_2$ subject to $x_1 + x_2 \leq m$ and $x_1, x_2 \geq 0$

$$\log(x_1) + x_2 - \lambda(x_1 + x_2 - m) + \mu_1(x_1) + \mu_2(x_2)$$

$$\frac{\partial(\log(x_1) + x_2 - \lambda(x_1 + x_2 - m) + \mu_1(x_1) + \mu_2(x_2))}{\partial x_1} = -\lambda + \mu_1 + \frac{1}{x_1}$$

$$\frac{\partial(\log(x_1) + x_2 - \lambda(x_1 + x_2 - m) + \mu_1(x_1) + \mu_2(x_2))}{\partial x_2} = -\lambda + \mu_2 + 1$$

FOCs

$$-\lambda + \mu_1 + \frac{1}{x_1} = 0$$

$$-\lambda + \mu_2 + 1 = 0$$

Three possible boundaries to be on. $x_1 \geq 0$, $x_2 \geq 0, x_1 + x_2 \leq m$.

We know $x_1 + x_2 = m$ at the optimum because the utility function is monotonic.

On the boundary of all three. $x_1 = 0, x_2 = 0, x_1 + x_2 = m$

This is impossible because there is no x_1, x_2 meeting these conditions.

On the boundary of $x_1 \geq 0$ and the budget constraint. $x_1 = 0, x_1 + x_2 = m$

This is a fesible scenario. $x_2 = m$. Set $\mu_2 = 0$ because we are on the interior of the $x_2 \geq 0$

$$-\lambda + \mu_1 + \frac{1}{x_1} = 0$$

$$-\lambda + 0 + 1 = 0$$

Let's simplify:

$$\mu_1 + \frac{1}{0} = 1$$

$$1 = \lambda$$

This is impossible. We cannot meet the first order conditions in this scenario.

Suppose $x_2 = 0$ and $x_1 + x_2 = m$ but $x_1 \geq 0$. Is there are bundle that meets this condition?

$$x_1 = m, x_2 = 0$$

$$-\lambda + \frac{1}{x_1} = 0$$

$$-\lambda + \mu_2 + 1 = 0$$

Plug in values:

$$\frac{1}{m} = \lambda$$

$$\mu_2 = \frac{1}{m} - 1$$

This is positive for $m \leq 1$. When $m \leq 1$ the optimal solution is $(m, 0)$.

Let's check the condition where both $x_1 > 0$ and $x_2 > 0$.

$$-\lambda + \frac{1}{x_1} = 0$$

$$-\lambda + 1 = 0$$

$$x_1 + x_2 = m$$

Solve these:

$$\lambda = 1$$

$$x_1 = 1$$

$$x_2 = m - 1$$

$$(1, m - 1)$$

Only feasible when $m \geq 1$

1.3 The marginal utility per dollar of all goods is the same for goods I buy some of at the optimum.

$$u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - m) + \mu_1(x_1) + \mu_2(x_2)$$

Suppose none of the non-negativity constraints bind. Then the first order conditions are:

$$\frac{MU_1}{p_1} = \lambda$$

$$\frac{MU_2}{p_2} = \lambda$$

Thus,

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

For a problem with more variables, this will also hold for and i, j.

$$\frac{MU_i}{p_i} = \frac{MU_j}{p_j}$$