1 Consumer Problem Continued

1.1 Utility Max

$x_1 x_2$, $p_1 x_1 + p_2 x_2 \leq m$
Marshallian Demands.

$$(x_1^*, x_2^*) = \left(\frac{\frac{1}{2}m}{p_1}, \frac{\frac{1}{2}m}{p_2} \right)$$

$$\lambda = \frac{m}{2p_1 p_2}$$

What is the maximum utility given the prices and income? **Indirect Utility.**

$$u\left(x_1^*, x_2^*\right)$$

$$V\left(p_1, p_2, m\right) = \frac{\frac{1}{2}m}{p_1}\frac{\frac{1}{2}m}{p_2} = \frac{\frac{1}{4}m^2}{p_1 p_2}$$

$$\frac{\frac{1}{4}m^2}{p_1 p_2}$$

$$\frac{\partial\left(\frac{\frac{1}{4}m^2}{p_1 p_2}\right)}{\partial p_1} \quad = \quad -\frac{m^2}{4p_1^2 p_2}$$

$$\frac{\frac{1}{4}\left(2m\right)^2}{\left(2p_1\right)\left(2p_2\right)} = \frac{\frac{1}{4}m^2}{p_1 p_2}$$

$$\frac{\frac{1}{4}m^2}{p_1 p_2}$$

A function that is homogenous of degree α has the following property:

$$f\left(tx\right) = t^{\alpha} f\left(x\right)$$

$$f(tx) = (tx)^2 = t^2x^2 = t^2 f(x)$$

$$2x_1 2x_2 = 4x_1x_2$$

Continuous- Due to **Berge's** Theorem of the Maximum

Homogenous of degree zero in prices and income.

Increasing in m and decreasing in $p_1, p_2, ...$

Quasi-convex in p, m- **If I take a budget that is a convex combination of two other budgets, the utility I can achieve cannot be better the best of the two budgets.**

$$(p_1, p_2, m) : (4, 2, 20), (2, 4, 20)$$

$$(3, 3, 20)$$

Envelope Condition:

$$(x_1x_2) - \lambda(p_1x_1 + p_2x_2 - m)$$

$\frac{\partial v}{\partial p}$

$$\frac{\partial((x_1x_2) - \lambda(p_1x_1 + p_2x_2 - m))}{\partial p_1}$$

$$-\lambda x_1$$

$$\lambda$$

$$\frac{\partial V(p_1, p_2, m)}{\partial p_1} = -\lambda x_1$$

$$\frac{\partial V(p_1, p_2, m)}{\partial m} = \lambda$$

$$-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial m}} = -\frac{-\lambda x_1}{\lambda} = x_1$$

1.2 Properties of Indirect Utility

For U that is continuous and strictly increasing, the Indirect Utility Function v has the Following Properties:

1. Continuous.
2. Homogeneous of degree zero in prices and income.
3. Strictly increasing in m and weakly decreasing in p.
4. Quasi-convex in (p, m).
5. Roy's Identity. $-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial m}} = x_i^*$ (An envelope condition.)

1.3 Cost Min Example

Minimize the cost of utility u with $x_1 x_2$

Min $p_1 x_1 + p_2 x_2$ subject to $x_1 x_2 \geq u$

$$\mathscr{L} = (p_1 x_1 + p_2 x_2) + \mu (u - x_1 x_2)$$

$$\frac{\partial ((p_1 x_1 + p_2 x_2) + \mu (u - x_1 x_2))}{\partial \mu} = u - x_1 x_2$$

$$\frac{\partial ((p_1 x_1 + p_2 x_2) + \mu (u - x_1 x_2))}{\partial x_1} = p_1 - \mu x_2$$

$$\frac{\partial ((p_1 x_1 + p_2 x_2) + \mu (u - x_1 x_2))}{\partial x_2} = p_2 - \mu x_1$$

$$u = x_1 x_2$$

$$x_1 = \frac{\sqrt{p_2}\sqrt{u}}{\sqrt{p_1}}, x_2 = \frac{\sqrt{p_1}\sqrt{u}}{\sqrt{p_2}}, \mu = \frac{\sqrt{p_1}\sqrt{p_2}}{\sqrt{u}}$$

Hicksian demands. What are the x_1, x_2 we need to minimize the cost of achieving utility u.

$$x_1^h = \frac{\sqrt{p_2}\sqrt{u}}{\sqrt{p_1}}$$

$$x_2^h = \frac{\sqrt{p_1}\sqrt{u}}{\sqrt{p_2}}$$

Pick an income m. Solve for $V(p_1, p_2, m)$. Then $x_1^h(p_1, p_2, V(p_1, p_2, m)) = x_1(p_1, p_2, m)$

Duality:

$$x_1^h(p_1, p_2, V(p_1, p_2, m)) = x_1(p_1, p_2, m)$$

$x_1 = \frac{\frac{1}{2}m}{p_1}, V(p_1, p_2, m) = \frac{\frac{1}{2}m}{p_1}\frac{\frac{1}{2}m}{p_2}, x_1^h = \frac{\sqrt{p_2}\sqrt{u}}{\sqrt{p_1}}$

$$\frac{\sqrt{p_2}\sqrt{\frac{\frac{1}{2}m}{p_1}\frac{\frac{1}{2}m}{p_2}}}{\sqrt{p_1}} = \frac{\frac{1}{2}m}{p_1}$$

$$\frac{\sqrt{p_2}\frac{1}{2}m\sqrt{\frac{1}{p_1}\frac{1}{p_2}}}{\sqrt{p_1}} = \frac{\frac{1}{2}m}{p_1}$$

$$\frac{\sqrt{p_2}\frac{1}{2}m}{\sqrt{p_1}\sqrt{p_1}\sqrt{p_2}} = \frac{\frac{1}{2}m}{p_1}$$

$$\frac{\frac{1}{2}m}{p_1} = \frac{\frac{1}{2}m}{p_1}$$

Another form of duality:

$$x_1^h(p_1, p_2, u) = x_1(p_1, p_2, E(p_1, p_2, u))$$

$$\frac{\sqrt{p_2}\sqrt{u}}{\sqrt{p_1}} = \frac{\frac{1}{2}E(p_1, p_2, u)}{p_2}$$

Here we need the "Expenditure function" this is the value of $p_1x_1 + p_2x_2$ evaluated at the hicksian demands.

$$E(p_1, p_2, u) = p_1\frac{\sqrt{p_2}\sqrt{u}}{\sqrt{p_1}} + p_2\frac{\sqrt{p_1}\sqrt{u}}{\sqrt{p_2}}$$

$$= 2\sqrt{p_1}\sqrt{p_2}\sqrt{u}$$

1.4 Properties of Expenditure Function

For U that is continuous and strictly increasing, the Expenditure Function e has the following properties:

1. Continuous.
2. For $p \gg 0$, strictly increasing and unbounded above in u.
3. Increasing in p.
4. Homogeneous of degree 1 in p.
5. Concave in p.
6. Shephard's lemma. *When x_i^h is single valued,* $-\frac{\partial e}{\partial p_i} = x_i^h$

1.5 Slutsky Equation

$$x_i\left(p, e\left(p, u\right)\right) = x_i^h\left(p, u\right)$$

$$\frac{\partial\left(x_i\left(p, e\left(p, u\right)\right)\right)}{\partial p_j} = \frac{\partial x_i^h\left(p, u\right)}{\partial p_j}$$

$$\frac{\partial\left(x_i\left(p, e\left(p, u\right)\right)\right)}{\partial p_j} = \frac{\partial x_i^h\left(p, u\right)}{\partial p_j}$$

$$\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j} + \frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y}\frac{\partial e}{\partial p_j} = \frac{\partial x_i^h\left(p, u\right)}{\partial p_j}$$

$$\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j} = \frac{\partial x_i^h\left(p, u\right)}{\partial p_j} - \frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y}x_j^h$$

1.5.1 Slutsky Equation: $\frac{\partial(x_i(p,y))}{\partial p_j} = \frac{\partial\left(x_i^h(p,\bar{u})\right)}{\partial p_j} - \frac{\partial(x_i(p,y))}{\partial y}x_j^h$.

1.5.2 Negative Own-Substitution Effects

1.5.3 Elasticity

Income Elasticity $\eta_i = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial y}{y}} = \frac{\partial x_i}{\partial y}\frac{y}{x_i}$

Price and Cross-Price Elasticity $\epsilon_{ij} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_j}{p_j}} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}$.

1.5.4 Elasticity Relations

The share-weighted elasticities with respect to good i is the negative of $i's$ share: $-s_i = \sum_{j=1}^{n} s_j \varepsilon_{j,i}$
The share-weighted income elasticities sum to 1: $1 = \sum_{j \in I} s_j \eta_j$

2 More Complex Optimization Examples

2.1 Some Examples with Multiple Constraints

Maximize $x_1 x_2$ subject to (1) $\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} \leq 10$ and (2) $2x_1 + x_2 \leq 40$.

Maximize $x_1 x_2$ subject to (1) $\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} \leq 10$, (2) $2x_1 + x_2 \leq 15$.