0.1 Utility Representation

Let $\succsim$ be the preference relation on $\mathcal{G}$.

$\succsim$ **is Complete.**

$\succsim$ **is transitive.**

$\succsim$ **is monotonicity.**

Since there are a finite number of events $a_1, a_2, ..., a_n$.

$$a_1 \succ a_2 \succ a_3 \succ ... \succ a_n$$

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n) \text{ if and only if } \alpha \geq \beta.$$

$\succsim$ **is continuous.**

For all $g \in \mathcal{G}$, there exists some $\alpha \in [0,1]$ such that $g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$

If preferences are complete, transitive, monotonic, and continuous there exists a utility function that represented $\succsim$.

$u(g)$ is the number that makes this true:

$$g \sim (u(g) \circ a_1, (1-u(g)) \circ a_n)$$

Suppose we have $A = \{0, 5, 10\}$. $10 \succ 5 \succ 0$

$$g_1 = (1 \circ 10).$$

$$u(g_1) = 1$$

$$g_2 = (1 \circ 0)$$

$$u(g_2) = 0$$

$$g_3 = (1 \circ 5).$$

Suppose by the archemedian prorperty that there is some probability over the best and worst outcome this is indifferent to. Suppose it is 0.5:

$$g_3 \sim (0.5 \circ 10, 0.5 \circ 0)$$

$$u(g_3) = 0.5$$

Finally:

$$g_4 = \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ (1 \circ \$5)\right)$$

There is some α such that

$$g_4 \sim (\alpha \circ 10, (1 - \alpha) \circ 0)$$

$$u(g_4) = \alpha$$

By continuity this number exists. **Utility Representation.** Under axioms $1 - 4$, we can represent $\succsim$ with a utility function. $u(g)$ is the number that makes this true:

$$g \sim (u(g) \circ a_1, (1 - u(g)) \circ a_n)$$

$$g \succsim g' \Leftrightarrow u(g) \geq u(g')$$

Start with

$$g \succsim g'$$

By **Continuity**

$$(u(g) \circ a_1, (1 - u(g) \circ a_n)) \sim g \succsim g' \sim (u(g') \circ a_1, (1 - u(g') \circ a_n))$$

By **transitivity**

$$g \succsim g' \Leftrightarrow (u(g) \circ a_1, (1 - u(g) \circ a_n)) \succsim (u(g') \circ a_1, (1 - u(g') \circ a_n))$$

By **monotonicity t**his is true if and only if $u(g) \geq u(g')$

$$g \succsim g' \Leftrightarrow u(g) \geq u(g')$$

We need two additional assumptions:

Substitutibility.

If we take a compound gamble and replace every gamble in it with an indifferent gamble, the result will be indiffernt to the original compound gamble.

$g_i \sim h_i$ for all $i \in \{1, ..., k\}$:

$$(p_1 \circ g_1, p_2 \circ g_2, p_3 \circ g_3, ..., p_k \circ g_k) \sim (p_1 \circ h_1, p_2 \circ h_2, p_3 \circ h_3, ..., p_k \circ h_k)$$

Example suppose this is true:

$$\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right)\sim(1\circ\$5)$$

From this compound gamble:

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\right)\sim\left(\frac{1}{2}\circ(1\circ\$5),\frac{1}{2}\circ(1\circ\$5)\right)$$

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\right)\sim\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right)\right)$$

Reduction.
Every gamble is indifferent to its induced simple gamble.

$$g\sim g_s(g)$$

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\right),\frac{1}{2}\circ\$5\right)\sim\frac{1}{8}\circ\$10,\frac{1}{8}\circ\$0,\frac{3}{4}\circ\$5$$

Also by reduction we know:

$$\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\sim\frac{1}{4}\circ\$10,\frac{1}{4}\circ\$0,\frac{1}{2}\circ\$5$$

If we have substitution:

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\right),\frac{1}{2}\circ\$5\right)\sim\left(\frac{1}{2}\circ\left(\frac{1}{4}\circ\$10,\frac{1}{4}\circ\$0,\frac{1}{2}\circ\$5\right),\frac{1}{2}\circ\$5\right)$$

If we don't have substitution we could have:

$$\left(\frac{1}{2}\circ\left(\frac{1}{4}\circ\$10,\frac{1}{4}\circ\$0,\frac{1}{2}\circ\$5\right),\frac{1}{2}\circ\$5\right)\succ\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\right),\frac{1}{2}\circ\$5\right)$$

0.2 Example of What Can Be Done with Subs. and Reduction

Suppose we have all of the assumptions:

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\right)$$

I want to turn this into a gamble over the best and worst outcome.
First let's turn it into a simple gamble:

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\right)\sim\frac{1}{4}\circ 10,\frac{1}{4}\circ 0,\frac{1}{2}\circ 5$$

From continuity $(1\circ 5)\sim(p_{0.5}\circ 10,(1-p_{0.5})\circ 0)$.
By substitution:

$$\frac{1}{4}\circ 10,\frac{1}{4}\circ 0,\frac{1}{2}\circ(p_{0.5}\circ 10,(1-p_{0.5})\circ 0)$$

By reduction this indifferent to:

$$\frac{1}{4}\circ 10,\frac{1}{4}\circ 0,\left(\frac{1}{2}p_{0.5}\circ 10,\frac{1}{2}(1-p_{0.5})\circ 0\right)$$

$$\left(\frac{1}{4}+\frac{1}{2}p_{0.5}\right)\circ 10,\frac{1}{4}+\frac{1}{2}(1-p_{0.5})\circ 0$$

Notice that $u(5)$ under our previous utility representation $5\sim(p_{0.5}\circ 10,(1-p_{0.5})\circ 0)$.
$u(5)=p_{0.5}$
Define the utility over outcomes to be the probaility that makes this true:

$$a_i\sim(u(a_i)\circ 10,(1-u(a_i))\circ 0)$$

$u(10)=1$,$u(5)=p_{0.5},u(0)=0$

$$\left(\frac{1}{4}+\frac{1}{2}p_{0.5}\right)=\frac{1}{4}(u(1))+\frac{1}{2}(u(5))+\frac{1}{4}(u(0))$$

By

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(p_{0.5}\circ\$10,(1-p_{0.5})\circ\$0)\right)$$

By reduction

$$\left(\frac{1}{4}+\frac{1}{2}p_{0.5}\circ\$10,\frac{1}{4}+\frac{1}{2}\left(1-p_{0.5}\right)\circ\$0\right)$$

$$\left(\frac{1}{4}1+\frac{1}{2}p_{0.5}+\frac{1}{4}0\right)\circ\$10,\left(\frac{1}{4}+\frac{1}{2}\left(1-p_{0.5}\right)\right)\circ\$0$$

$$\frac{1}{4}u\left(10\right)+\frac{1}{2}u\left(5\right)+\frac{1}{4}u\left(0\right)$$

$$\frac{1}{4}\circ 10,\frac{1}{2}\circ 5,\frac{1}{4}\circ 0$$

With subsitution and reduction not only does every gamble have some p such that

$$g\sim\left(p\circ 10,\left(1-p\right)\circ 0\right)$$

p must be a weighted sum of the utilities of the outcomes in the induced simple gamble to p.

As long as we have reduction, substitution, continutity, montonicity, transivity, and completness.

The utility over every gamble can be represented as a weighted sum of the utility the outcomes of its induced simple gamble where the utility of the outcomes are weighted by the probability in the induced simple gamble.

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ\left(1\circ\$5\right)\right)\sim\frac{1}{4}\circ 10,\frac{1}{4}\circ 0,\frac{1}{2}\circ 5$$

$u\left(10\right),u\left(5\right),u\left(0\right)$

$$u\left(g\right)=\frac{1}{4}u\left(10\right)+\frac{1}{2}u\left(5\right)+\frac{1}{4}u\left(0\right)$$

$u\left(w\right)=\log\left(w+1\right)$

$$u\left(g\right)=\frac{1}{4}\left(\log\left(10+1\right)\right)+\frac{1}{2}\left(\log\left(5+1\right)\right)+\frac{1}{4}\left(\log\left(0+1\right)\right)$$

$$\frac{1}{4}\left(\log\left(10+1\right)\right)+\frac{1}{2}\left(\log\left(5+1\right)\right)+\frac{1}{4}\left(\log\left(0+1.0\right)\right)\quad=\quad 1.49535$$

0.3 Theorem

Expected Utility Representation. There is a u with the *expected* utility property such that $u(g) \geq u(g') \Leftrightarrow g \succsim g'$ if and only if $\succsim$ meets axioms 1-6.

Let $u(g) = \sum_{i=1}^{n} p_i^g u(a_i)$.

$$u(10) = 100$$

$$u(5) = 5$$

$$u(0) = 0$$

0.4