0.1 A few Cost Functions From Last Class

$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$
Minimize cost of producing y

$$(w_1 x_1 + w_2 x_2) + \mu \left(y - x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \right)$$

$x_1 = \frac{\sqrt{w_2}}{\sqrt{w_1}} y, x_2 = \frac{\sqrt{w_1}}{\sqrt{w_2}} y, c(y) = 2\sqrt{w_1 w_2} y$

$$c(1) = 2\sqrt{w_1 w_2}$$

$$2\sqrt{w_1 w_2} y = c(1) y$$

Conditional factor demands and the cost function:
$f(x_1, x_2) = \log(x_1) + \log(x_2)$

$$(w_1 * x_1 + w_2 * x_2) + \mu (y - \log(x_1) - \log(x_2))$$

$x_1 \to \frac{\sqrt{w_2} e^{y/2}}{\sqrt{w_1}}, x_2 \to \frac{\sqrt{w_1} e^{y/2}}{\sqrt{w_2}}, c(y) = 2\sqrt{w_1 w_2} e^{\frac{y}{2}}$
Notice that the cost function is not quite in the form $c(1) f(y)$ since:

$$c(1) = 2\sqrt{w_1 w_2} e^{\frac{1}{2}}$$

To put this in the form $c(y) = c(1) f(y)$ we need:

$$e^{\frac{y-1}{2}} = f(y)$$

Now it is in the right form:

$$c(y) = \left(2\sqrt{w_1 w_2} e^{\frac{1}{2}} \right) e^{\frac{y-1}{2}}$$

0.2 A Separable Production Problem

Let's suppose $w_1 = w_2 = w_3 = w_4 = 1$

$$f(x_1, x_2, x_3, x_4) = \log(x_1) + \log(x_2) + x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}$$

$$f(x_1, x_2, x_3, x_4) = f_1(x_1, x_2) + f_2(x_3, x_4)$$

Weakly separable into the groups (x_1, x_2) and (x_3, x_4).

To see if a function can be separated into groups. Check whether ratio of any two marginal products from a group depend only on variables from that group.

$$\log(x_1) + \log(x_2) + x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}$$

$$\frac{\frac{\partial(f)}{\partial x_1}}{\frac{\partial(f)}{\partial x_2}} = \frac{\frac{1}{x_1}}{\frac{1}{x_2}} = \frac{x_2}{x_1}$$

$$\frac{\frac{\partial(f)}{\partial x_3}}{\frac{\partial(f)}{\partial x_3}} = \frac{x_4}{x_3}$$

While this is weakly separable into groups (x_1, x_2) and (x_3, x_4). It is not strongly separable which would require it be separable into any groups.

To see that we can't make x_2, x_3 a group note that the ration of their marginal products depends on x_4.

$$\frac{\frac{\partial\left(\log(x_1)+\log(x_2)+x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)}{\partial x_2}}{\frac{\partial\left(\log(x_1)+\log(x_2)+x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)}{\partial x_3}} = \frac{2\sqrt{x_3}}{x_2\sqrt{x_4}}$$

This production function is strongly separable:

$$x_1 x_2 x_3 x_4$$

For instance, x_1 and x_2 can be a group:

$$\frac{\frac{\partial(f)}{\partial x_1}}{\frac{\partial(f)}{\partial x_2}} = \frac{x_2 x_3 x_4}{x_1 x_3 x_4} = \frac{x_2}{x_1}$$

Any other pair of inputs would have a similar ratio of marginal products only depending on those two. Thus, this production function is **strongly separable**.

0.3 Separable Production

Let's use separability to help solve this cost minimization problem:

Let's suppose $w_1 = w_2 = w_3 = w_4 = 1$

$$f(x_1, x_2, x_3, x_4) = \log(x_1) + \log(x_2) + x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}$$

Rewrite the production using "intermediate goods":

$$f(y_1, y_2) = y_1 + y_2$$

Minimize the cost of producing y using y_1, y_2. We already know the cost of producing y_1 and y_2 in an optimal way from our previous work on these production functions:

Recall that for $x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ the cost function is $c_1(y_2) = 2y_2$.

Recall that for $\log(x_1) + \log(x_2)$ the cost function is $c_2(y_1) = 2e^{\frac{y_1}{2}}$.

Thus we can write the overall cost function for y: $c(y) = c_1(y_1) + c_2(y_2) = 2e^{\frac{y_1}{2}} + 2y_2$

We now minimize this cost subject to the production constraint $y = y_1 + y_2$

$$\left(2e^{\frac{y_1}{2}} + 2y_2\right) + \mu(y - y_1 - y_2)$$

$$\frac{\partial\left(\left(2e^{\frac{y_1}{2}} + 2y_2\right) + \mu(y - y_1 - y_2)\right)}{\partial y_1} = e^{\frac{y_1}{2}} - \mu$$

$$\frac{\partial\left(\left(2e^{\frac{y_1}{2}} + 2y_2\right) + \mu(y - y_1 - y_2)\right)}{\partial y_2} = 2 - \mu$$

Combining these:

$$e^{\frac{y_1}{2}} = 2$$

Solve to get the optimal y_1:

$$\frac{y_1}{2} = 0.693147$$

$$y_1 = 1.38629$$

Now use the production constraint to get y_2:

$$y = (1.38629) + y_2$$

$$y - 1.38629 = y_2$$

If $y < 1.38629$ then $y_1 = y$.

If $y \geq 1.38629$ then $y_1 = 1.38629$ and $y_2 = y - 1.38629$.

We can use the conditional factor demands to get the optimal x_1, x_2, x_3, x_4. For instance, suppose that $y \geq 1.38629$ then $y_1 = 1.38629$. Find the optimal x_1, x_2 using our original conditional factor demands:

$$x_1 = e^{(1.38629)/2}, x_2 = e^{(1.38629)/2}$$

1 Profit

We think of firms as agents that maximize profit.

$$\pi = revenue - cost$$

In terms of inputs we can write profit function as (where $p()$ is the price the firm can get for selling output $f(x_1, x_2)$:

$$\pi(x_1, x_2) = f(x_1, x_2)\, p(f(x_1, x_2)) - (w_1 x_1 + w_2 x_2)$$

Suppose that price is fixed (price-taking assumption). This is a simplifying assumption that is only appropriate when the firm is a *very small* part of the overall market. **Perfect Competition.** In this case $p()$ is just a constant p and the profit function is:

$$\pi(x_1, x_2) = f(x_1, x_2)\, p - (w_1 x_1 + w_2 x_2)$$

Let's suppose production function $x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$, $w_1 = w_2 = 1$

$$\pi(x_1, x_2) = p * x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} - (x_1 + x_2)$$

Maximize this:

$$\frac{\partial\left(p * x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} - (x_1 + x_2)\right)}{\partial x_1} = \frac{p\sqrt[4]{x_2}}{4x_1^{3/4}} - 1$$

$$\frac{\partial\left(p * x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} - (x_1 + x_2)\right)}{\partial x_2} = \frac{p\sqrt[4]{x_1}}{4x_2^{3/4}} - 1$$

$$x_1 \to \frac{p^2}{16}, x_2 \to \frac{p^2}{16}$$

Profit maximization implies cost minimization.

$$\pi(q) = p(q)\, q - c(q)$$

Because profit max implies cost minimization, we can rewrite the profit function purely in terms of output q:

Let's suppose we have $q = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$ and $w_1 = w_2 = 1$. For this production function the cost function is:

$$c(q) = 2q^2$$

The profit function purely in terms of q in perfect competition:

$$\pi(q) = pq - 2q^2$$

Now we can maximize this by finding the one-dimensional first order condition:

$$\frac{\partial\left(p * q - 2q^2\right)}{\partial q} = p - 4q$$

$$q = \frac{1}{4}p$$

Now we can work backwards to get the inputs by plugging the optimal output into the conditional factor demands for x_1 and x_2 which in this case would give us:

$$x_1 = x_2 = \frac{1}{16}p^2$$

1.1 Linear Cost

Writing the profit function in terms of q makes it easy to see when there is no profit maximizing level output or when the profit maximizing level is 0. For instance, suppose:
$f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, and $w_1 = w_2 = 1$.
The profit function in terms of q is:

$$\pi(q) = pq - 2q = (p - 2)\, q$$

Thus, if $p < 2$ then the optimal is $q = 0$ and if $p > 2$ there is no optimal output.

1.2 Example 2

Let $w_1 = w_2 = 1$. Suppose a firm's production function is $f(x_1, x_2) = \log(x_1) + \log(x_2)$.

$$c(y) = 2e^{\frac{y}{2}}$$

Profit:

$$p * y - 2e^{\frac{y}{2}}$$

Maximize this:

$$\frac{\partial\left(p * y - 2e^{\frac{y}{2}}\right)}{\partial y} = p - e^{y/2}$$

$$p = e^{y/2}$$

$$y = 2\log(p)$$

1.3 The Profit Function

The "profit function" is analogous to the cost function and the profit of the firm evaluated at the optimal q^* and cost minimized inputs. It is $\pi(p,w) = max_q pq - c(q) = Max_x\, pq - (w_1x_1 + w_2x_2)$. **When well defined** and under the assumption of **perfect competition** it has the following properties:

Properties of the profit function in perfect competition:

1. Increasing in p,
2. Decreasing in w,
3. Homogeneous of degree one in p, w
4. Convex in p, w (why)
5. Hotelling: $\frac{\partial\pi}{\partial p} = q(p,w), -\frac{\partial\pi}{\partial w_i} = x_i(p,w)$

Combining 4 and 5 we can prove that, output is increasing in price (weakly), and any input is weakly decreasing in it's own wage (this is the substitution effect for production) $\frac{\partial q}{\partial p} \geq 0$, $\frac{\partial x_i}{\partial w_i} \leq 0$.