Part I

Markets feat. Cournot

1 The Cournot Oligopoly Model

Under price taking:

$$\pi(q) = qp - c(q)$$

What should we do with p if we relax this assumption?

$$\pi(q) = qp(q) - c(q)$$

If this firm was a monopoly, the could charge as much as consumers are willing to pay to buy q units. *Inverse demand.*
A demand function:

$$q = \frac{\frac{1}{2}M}{p}$$

Inverse demand:

$$p = \frac{\frac{1}{2}M}{q}$$

If a firm was a monopolist facing this demand function:

$$\pi(q) = q\frac{\frac{1}{2}M}{q} - c(q)$$

Suppose $c(q) = q^2$

$$\pi(q) = q\frac{\frac{1}{2}M}{q} - q^2$$

$$\pi(q) = \frac{1}{2}M - q^2$$

1.1 Another Monopoly Problem

$q(p) = 100 - p$. Inverse demand $p(q) = 100 - q$. Let's assume $c(q) = q^2$.

$$\pi(q) = q(100 - q) - q^2$$

$$\frac{\partial \left(q(100 - q) - q^2\right)}{\partial q} = 100 - 4q$$

$$q = 25$$

$$p(q) = 100 - p = 100 - 25 = 75$$

$$\pi(q) = 25(75) - 25^2 = 1250$$

1.2 Cournot Oligopoly

There are two firms with the same cost function $c(q) = 2q^2$ market demand is $Q = \frac{100}{p}$. Inverse demand $p = \frac{100}{Q}$.

$$\pi_1(q_1, q_2) = q_1 \frac{100}{q_1 + q_2} - 2q_1^2$$

$$\frac{\partial \left(q_1 \frac{100}{q_1+q_2} - 2q_1^2\right)}{\partial q_1} = -\frac{100q_1}{(q_1 + q_2)^2} - 4q_1 + \frac{100}{q_1 + q_2}$$

$$-\frac{100q_1}{(q_1 + q_2)^2} - 4q_1 + \frac{100}{q_1 + q_2} = 0$$

Compare this to perfect competition first order condition:

$$pq - c(q)$$

$$p - mc(q) = 0$$

$$-\frac{100q_1}{(q_1 + q_2)^2} + \frac{100}{q_1 + q_2} - 4q_1 = 0$$

The new indirect term $-\frac{100q_1}{(q_1+q_2)^2}$ is accounting for how price changes when I change my quantity.

1.3 Simpler Cournot

$c(q) = q^2$. $p(Q) = 100 - Q$

$$\pi_1(q_1, q_2) = q_1(100 - (q_1 + q_2)) - q_1^2$$

$$\frac{\partial\left(q_1(100 - (q_1 + q_2)) - q_1^2\right)}{\partial q_1} = -4q_1 - q_2 + 100$$

$$Solve[-4q_1 - q_2 + 100 == 0]$$

$$q_1 = \frac{1}{4}(100 - q_2)$$

This is what we call a **best response function.**

$$q_2 = \frac{1}{4}(100 - q_1)$$

Suppose firm one knows$q_2 = 30$. What would firm 1 like to do?

$$\pi_1(q_1, 30) = q_1(100 - (q_1 + 30)) - q_1^2$$

$$\frac{\partial\left(q_1(100 - (q_1 + 30)) - q_1^2\right)}{\partial q_1} = 70 - 4q_1$$

$$q_1 \to \frac{35}{2}$$

This is the same as we get by plugging $q_2 = 30$ into the best response:

$$q_1 = \frac{1}{4}(100 - 30)$$

$$q_1 = \frac{35}{2}$$

What would firm 2 like to do?

$$q_2 = \frac{1}{4}\left(100 - \frac{35}{2}\right) = \frac{165}{8} \neq 30$$

Given the set of choices $\left(\frac{35}{2}, 30\right)$ some firm (in this case firm 2) has incentive to change what they are doing.

To find a profile of choices where this doesn't happen, solve the best response functions simultaneously.

$$q_1 = \frac{1}{4}\left(100 - q_2\right)$$

$$q_2 = \frac{1}{4}\left(100 - q_1\right)$$

$$Solve[\{q_1 = \frac{1}{4}\left(100 - q_2\right), q_2 = \frac{1}{4}\left(100 - q_1\right)\}, \{q_1, q_2\}]$$

$$\{\{q_1 \to 20, q_2 \to 20\}\}$$

We have a name for a situation where profile of choices are mutually best responses. **Nash Equilibrium.**

A symmetric nash equilibrium is a $q = q_1 = q_2$ that solves:

$$q = \frac{1}{4}\left(100 - q\right)$$

$$q = \frac{1}{4}\left(100 - q\right)$$

When we want to solve for symmetric eq. we can impose symmetry on the best response of any of the firms and solve the resulting equation:

$$Solve[q = \frac{1}{4}\left(100 - q\right)]$$

$$q = 20$$

1.4 What's a Game?

A **game** is a set $\Gamma = \{P, S, \pi\}$. P is the set of players, S is the set of sets of strategies for each player, π are the payoffs functions mapping S into $\mathbb{R}^n$.

Players are firm 1 and firm 2: $P = \{1, 2\}$

$S = (\mathbb{R}_+, \mathbb{R}_+)$. `Player 1 chooses a quantity` $q_1 \in \mathbb{R}_+$. `Player 2 chooses a quantity` $q_2 \in \mathbb{R}_+$.

Payoffs. π_1 profit and π_2.

The nash equilibrium of a game is a set of strategies that are mutually best responses. They are strategies that maximize each player's profit conditional on the strategies of the others.

1.5 Cournot Oligopoly with N firms:

$c(q) = q^2$. $p(Q) = 100 - Q$. Let $Q_{-i} = Q - q_i$

$$\pi_i (q_i, Q_{-i}) = q_i (100 - (q_i + Q_{-i})) - q_i^2$$

$$\frac{\partial \left(q_i (100 - (q_i + Q_{-i})) - q_i^2\right)}{\partial q_i} = -4q_i - Q_{-i} + 100$$

The best response function:

$$q_i = \frac{1}{4}(100 - Q_{-i})$$

If we want to find a symmetric equilibrium. Impose symmetry:

$$q = \frac{1}{4}(100 - (N-1)q)$$

The Nash equilibrium. Each firm produces $q^*(N)$

$$q^*(N) = \frac{100}{N+3}$$

Let's look at some special cases:

$$q^*(2) = \frac{100}{2+3} = 20$$

$$q^*(1) = \frac{100}{1+3} = 25$$

$$lim_{N \to \infty} q^*(N) = 0$$

Market Quantity:

$$Q^*(N) = N\frac{100}{N+3}$$

Market Price in Equilibrium:

$$p^*(N) = 100 - Q = 100 - N\frac{100}{N+3}$$

Profit of each firm:

$$\pi\left(q^*\left(N\right)\right) = \frac{100}{N+3}\left(100 - N\frac{100}{N+3}\right) - \frac{100}{N+3}^{2}$$

$$\begin{pmatrix} 1 & 75. & 25. & 1250. \\ 2 & 60. & 40. & 800. \\ 5 & 37.5 & 62.5 & 312.5 \\ 10 & 23.0769 & 76.9231 & 118.343 \\ 100 & 2.91262 & 97.0874 & 1.88519 \\ 1000 & 0.299103 & 99.7009 & 0.0198805 \end{pmatrix}$$

1.6 Equilibrium

The **best response correspondence** $B_i\left(S_{-i}\right)$ for player i is given by $B_i\left(S_{-i}\right) = \{s_i | s_i \in S_i \, s_i \in arg.max_{S_i} \pi_i\left(s_i, S_{-i}\right)\}$. A Nash equilibrium requires that all players are best responding to each-others strategies. A strategy profile $s = \left(s_1, ..., s_n\right)$ is called a **Nash equilibrium** if for all $i \in \{1, ..., n\}$, $s_i \in B\left(S_{-i}\right)$.

1.7 J Firms

1.8 How Reasonable is Price Taking?

1.9 Solved Examples

1.9.1 *Entry With Cournot.*

Suppose it costs $1 to enter this market.

1.9.2 *Asymmetric Cournot*

Asymmetric Cournot. Suppose there are now two firms with different cost functions: $c_1\left(q_1\right) = 2q_1^2$, $c_2\left(q_2\right) = 2q_2^{\frac{3}{2}}$. What are the quantities of the two firms in Nash equilibrium?

1.10 Stackleberg Model of Sequential Quantity-Setting

Stackleberg Model Suppose cost is $2q^2$ with inverse demand of $p = 100 - Q$. Firm one moves first, choosing q_1. Firm 2 observes q_1 and then chooses q_2. What are q_1 and q_2 in equilibrium?

1.11 Colluding Firms

Let's stick with demand of $100 - p$ and and cost $2q^2$ for each firm.