0.1 Expected Utility Redux

0.1.1 Representing $\succsim$ with utility function.

Completeness, Transitivity, **Continuity (every gamble is indifferent to some gamble over the best and worst outcome), Monotonicity (Gambles over the best and worst outcome that put a strictly higher probability on the best outcome are strictly better).**

$$g \sim (\alpha \circ a_1, (1-\alpha) \circ a_n)$$

Suppose $g \succsim g'$

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \sim g \succsim g' \sim (\alpha' \circ a_1, (1-\alpha') \circ a_n)$$

$$(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\alpha' \circ a_1, (1-\alpha') \circ a_n)$$

$$\alpha \geq \alpha'$$

0.1.2 Linear structure of utility function.

Under two additional assumptions.

Substitution (in any compound gamble, we can replace some subgamble with and indifferent one, and the result is indifferent to the original compound gamble), Reduction (every gamble is indifferent to its induced simple gamble)

Let's suppose we have $a = \{0, 5, 10\}$.

Because of continuity we can turn each of these outcomes into a gamble over the best and worst outcome.

$$10 \sim (1 \circ 10, 0 \circ 0)$$

$$\alpha_{10} = 1$$

$$5 \sim (\alpha_5 \circ 10, (1-\alpha_5) \circ 0)$$

$$\alpha_5$$

$$0 \sim (0 \circ 10, 1 \circ 0)$$

$$\alpha_0 = 0$$

$$g = \left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{3}{4} \circ 10, \frac{1}{4} \circ 0\right), \frac{1}{2} \circ 5\right)\right), \left(\frac{1}{2} \circ 5\right)\right)$$

By reduction $g \sim g_1$

$$g_1 = \left(\frac{3}{16} \circ 10, \frac{3}{4} \circ 5, \frac{1}{16} \circ 0\right)$$

By **continuity** and **substitution** we replace each outcome with a gamble over the best and worst outcome and we have $g_1 \sim g_2$.

$$g_2 = \left(\frac{3}{16} \circ (1 \circ 10, 0 \circ 0), \frac{3}{4} \circ (\alpha_5 \circ 10, (1 - \alpha_5) \circ 0), \frac{1}{16} \circ (0 \circ 10, 1 \circ 0)\right)$$

By reduction $g_2 \sim g_3$

$$g_3 = \left(\frac{3}{16}(1) + \frac{3}{4}(\alpha_5) + \frac{1}{16}(0) \circ 10, \left(1 - \frac{3}{16}(1) + \frac{3}{4}(\alpha_5) + \frac{1}{16}(0)\right) \circ 0\right)$$

By transitivity $g \sim g_3$

What probability of the best outcome is associated with g?

$$\alpha_g = \frac{3}{16}(\alpha_{10}) + \frac{3}{4}(\alpha_5) + \frac{1}{16}(\alpha_0)$$

Since preferences are continuous and monotonic, we know we can use $u(a) = \alpha_a$ to represent preferences. If we do this, then $u(g)$ **has to be the expected utility of the outcomes of that gamble.**

$$u(g) = \frac{3}{16}(1) + \frac{3}{4}(\alpha_5) + \frac{1}{16}(0)$$

$$u(g) = \frac{3}{16}u(10) + \frac{3}{4}u(5) + \frac{1}{16}u(0)$$

0.1.3 Transformations.

Suppose the consumer is an expected utility maximizer with utility over outcomes $\{0, 5, 10\}$ $u(0) = 0, u(5) = 0.5, u(10) = 1$.

$$g = \left(\frac{1}{2} \circ \left(\frac{3}{4} \circ 10, \frac{1}{4} \circ 0\right)\right), \left(\frac{1}{2} \circ 5\right)$$

$$\sim \left(\frac{3}{8} \circ 10, \frac{1}{2} \circ 5, \frac{1}{8} \circ 0\right)$$

$$u\left(g\right) = \frac{3}{8}u\left(10\right) + \frac{1}{2}u\left(5\right) + \frac{1}{8}u\left(0\right)$$

$$u\left(g\right) = \frac{3}{8}1 + \frac{1}{2}0.5 + \frac{1}{8}0$$

$$u\left(g\right) = 0.625$$

Now suppose we use an affine transformation.
$u' = 2u\left(g\right) + 1$
What if we apply this to $u\left(g\right) = 0.625$? $u'\left(g\right) = 2 * 0.625 + 1 = 2.25$
$u'\left(10\right) = 3, u'\left(5\right) = 2, u'\left(0\right) = 1$. Does the expected utility property hold?

$$u\left(g\right) = \frac{3}{8}\left(3\right) + \frac{1}{2}\left(2\right) + \frac{1}{8}\left(1\right) = 2.25$$

The expected utility property still holds.
Now suppose we use a non-affine transformation.
$u' = u\left(g\right)^2$
What if we apply this to $u\left(g\right) = 0.625$? $u'\left(g\right) = 0.625^2 = 0.390625$
$u'\left(10\right) = 1, u'\left(5\right) = 0.25, u'\left(0\right) = 0$. Does the expected utility property hold?

$$u'\left(g\right) = \frac{3}{8}\left(1\right) + \frac{1}{2}\left(0.25\right) + \frac{1}{8}\left(0\right) = 0.5$$

0.2 Example

Suppose a consumer is a expected utility maximizer with utility of wealth equal to: $v\left(w\right) = w^2$

$$v'\left(w\right) = 2w^2 + 5$$

v' retains expected utility property because

$$av\left(w\right) + b$$

If they satisfy $1 - 6$ there is some utility function with the expected utility property.
We still have to be told what it is.

0.3 Lexicographic Preferences over Gambles

A consumers preferences are that a gamble is better if it's induced simple gamble gives a higher probability to the best outcome. Otherwise, if they give the same probability to the best outcome, then it is prefered if it gives a higher probability to the second best outcome....

$$(0.8 \circ 10, 0.1 \circ 5, 0.1 \circ 0) \succ (0.7 \circ 10, 0.2 \circ 5, 0.1 \circ 0)$$

$$(0.8 \circ 10, 0.2 \circ 5, 0.0 \circ 0) \succ (0.8 \circ 10, 0.1 \circ 5, 0.1 \circ 0)$$

0.4 Two Measures of Risk Preferences and Differential Equations

If a consumers the utility of a gamble is less than the utility of the expected outcome of a gamble, we say the consumer is **risk averse**. There are several measures of "how" risk averse someone is.

Relative Risk Aversion:$-\frac{wv''(w)}{v'(w)}$

What is this?

How does something change in percentage terms when something else changes 1%?

Relative risk aversion tells us how much the marginal utility for wealth decreases and wealth goes up by 1%

$$\frac{\partial\left(v'\left(w\right)\right)}{\partial w}\frac{w}{v'\left(w\right)} = \frac{v''\left(w\right)w}{v'\left(w\right)}$$

$$\frac{v''\left(w\right)w}{v'\left(w\right)} = c$$

What are all the utility functions that give CRRA?

$$DSolve[w * v''[w]/v'[w] == c, v[w], w]$$

Mathematica gives us this:

$$\left\{\left\{v(w) \to \frac{c_1 w^{c+1}}{c+1} + c_2\right\}\right\}$$

This tells us all the CRRA utility functions are affine transformations of something like this (where c is the measure of constant relative risk aversion) by setting $c_1 = 1$ and $c_2 = 0$:

$$v\left(w\right) = \frac{w^{c+1}}{c+1}$$