0.1 Cournot

$c(q) = q_i^2$. Inverse demand $p(Q) = 100 - Q$

$$\pi(q_i, Q_{-i}) = q_i(100 - Q) - q_i^2$$

$$\pi(q_i, Q_{-i}) = q_i(100 - (q_i + Q_{-i})) - q_i^2$$

$$= 100q_i - q_i^2 - q_i Q_{-i} - q_i^2$$

$$= 100q_i - 2q_i^2 - q_i Q_{-i}$$

The firm maximizes quantity where the marginal profit is zero:

$$\frac{\partial\left(100q_i - 2q_i^2 - q_i Q_{-i}\right)}{\partial q_i} = -4q_i - Q_{-i} + 100$$

The firm maximizes profit:

$$-4q_i - Q_{-i} + 100 = 0$$

$$q_i = \frac{100 - Q_{-i}}{4} = 25 - \frac{1}{4}Q_{-i}$$

$$q_i = 25 - \frac{1}{4}Q_{-i}$$

0.2 Two firms

Best response:

$$B_1(q_2) = 25 - \frac{1}{4}q_2$$

$$B_2(q_1) = 25 - \frac{1}{4}q_1$$

In nash equilibrlim, we need a set of quantities that are mutual best responses.

q_1, q_2 such that $B_1(q_2) = q_1$ and $a = q_2$.

$$B(q) = q$$

This is another way of saying that we are looking for a strategy profile that is a best response to itself.

$$q_1 = 20, q_2 = 20$$

Let's check this is a fixed point:

$$q_1 = 25 - \frac{1}{4}20 = 25 - 5 = 20$$

$$q_2 = 25 - \frac{1}{4}20 = 25 - 5 = 20$$

$B(20, 20) = (20, 20)$

0.3 Fixed Points

$f(x) = \frac{1}{2}\sqrt{x} + 1$

To find the fixed points of this function, we look for $f(x) = x$

$$Solve[\frac{1}{2}\sqrt{x} + 1 == x]$$

$$\left\{\left\{x \to \frac{1}{8}\left(9 + \sqrt{17}\right)\right\}\right\}$$

$$1.64039$$

$$\frac{1}{2}\sqrt{\frac{1}{8}\left(9 + \sqrt{17}\right)} + 1.0 = 1.64039$$

$f(x) = x + 1$

$$x = x + 1$$

In Game theory, Kakutani's fixed point theorem.

Anothe example:

$f(x) = \frac{2}{3}\sqrt{x} + 1$

$$\left\{\left\{x \to \frac{1}{9}\left(11 + 2\sqrt{10}\right)\right\}\right\}$$

$$\frac{1}{9}\left(11 + 2.0\sqrt{10}\right) \quad = \quad 1.92495$$

Let's try $f(10) = \frac{2}{3}\sqrt{10} + 1.0 = 3.10819$

$f(3.10819) = 2.17534$

$f(2.17534) = 1.98327$

0.4 Contractions

A contraction is a function on some metric space (X, d) such that $f : X \to X$.

If there exists some $k \in [0, 1)$

$$\forall x, x' \in X, \quad d\left(f\left(x\right), f\left(x'\right)\right) \leq kd\left(x, x'\right)$$

There has so be some k strictly less than one such that the distance between two points after they passed throug the function is less than k times the distance of the original two points.

0.5 Banach Fixed-Point Theorem

Banach fixed-point theorem. Every contraction mapping on a non-empty complete metric space (X, d)has a unique fixed point $x^* \in X$ and $\forall x \in X$ the sequence $(x, f\left(x\right), f\left(f\left(x\right)\right), ...)$ converges to x^*. Let's prove $\frac{2}{3}\sqrt{x}+1$ is a contraction $[1, \infty)$

We pick an x and x'.

Let's use absolute distance between the points. Assume WLOG that $x > x'$.

$$\left(\frac{2}{3}\sqrt{x}+1-\left(\frac{2}{3}\sqrt{x'}+1\right)\right)$$

$$\frac{2}{3}\sqrt{x}+1-\frac{2}{3}\sqrt{x'}-1$$

$$\frac{2}{3}\left(\sqrt{x}-\sqrt{x'}\right) \leq \frac{2}{3}\left(x-x'\right)$$

Every two points will be mapped into two new points that are always less than $\frac{2}{3}$ times the distance of the original two points. Thus, this is a contraction on $[1, \infty)$.

0.6 Back to Cournot

$$B_1\left(q_2\right) = 25 - \frac{1}{4}q_2$$

$$B_2\left(q_1\right) = 25 - \frac{1}{4}q_1$$

$$\left(q_1, q_2\right), \left(q_1', q_2'\right)$$

Jacobian is a the matrix of partial derivative of the function:

$$J = \begin{matrix} 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{matrix}$$

If any of the norms of this Jacobian are uniformly bounded below 1 in some region where the function maps that region into itself, then the function is a contraction on that region. Here this is true for any region. We can use the l_1 norm which is simply the maximum of the the maximum of the column sums of the absolute values of the entries (see notes for more detail on matrix norms). The column sums of the absolute values of the entries are both $\frac{1}{4}$. Thus, this is a contraction!