0.1 Collusion with 2 Firms

$$\pi_{cournot} = 612.25$$

$$\pi_{collusion} = 625$$

$$\pi_{deviating} \approx 638.02$$

$$\pi_{non-deviating} \approx 598.96$$

Turn this into a infinitely repeated game. The players play the game every day forever. Discount each subsequent day by $\beta \in (0, 1)$.

Suppose the players play the curnot strategy every day:

$$612.25 + \beta 612.25 + \beta^2 612.25 + ... = 612.25 \sum_{t=0}^{\infty} \beta^t = \frac{612.25}{1-\beta}$$

If $\beta = \frac{1}{2}$:

$$1224.5$$

A strategy in a infinitely repeated game is a choice of action for every possible history of play.

A profile of strategies is a sub-game perfect Nash equilibrium of the infinitely repeated game if it is optimal for each player given the strategies of the other players.

Grim trigger strategy: Collude as long as the history of play only includes the collusive quantity or we are in first period of the game. Play Cournot quantity otherwise.

This game can only be in two states. "Good state" G includes only collusion. "Bad state" B includes something else.

We can write the value of being in a particular state and playing according to the strategy:

$$V(g) = 625 \sum_{t=0}^{\infty} \beta^t = \frac{625}{1-\beta}$$

$$V(b) = 612.25 \sum_{t=0}^{\infty} \beta^t = \frac{612.25}{1-\beta}$$

Principle of optimality (one-shot deviation principle). A strategy or plan is optimal if there there is no state in which a player has a payoff-improving deviation from plan, holding the rest of the plan fixed.

Check state g. If I follow the plan:

$$625 + \beta V(g)$$

If I deviate, then I will be in state b tomorrow, so I better maximize my payoff today.

$$638.02 + \beta V(b)$$

There is no one-shot deviation if going with the plan is better than the best one-shot deviation:

$$625 + \beta V(g) > 638.02 + \beta V(b)$$

$$\left(625 + \beta \frac{625}{1-\beta}\right) > \left(638.02 + \beta \frac{612.25}{1-\beta}\right)$$

$$0.505239 < \beta < 1.$$

As long as $\beta > 0.505239$ then there is no one-shot deviation from the good state. What about the bad state? There is no possible one-shot deviation because the player's strategies are already optimal at every stage given the other's strategy because it is a Nash equilibrium.

As long as $\beta > 0.505239$ collusion can be supported in subgame perfect nash equilibrium using the grim trigger strategy.

$$\beta \frac{625}{1-\beta} - \beta \frac{612.25}{1-\beta} > 638.02 - 625$$

$$\frac{\beta}{1-\beta} 12.25 > 13.02$$

0.2 Grim Trigger in the Prisoner's Dilemma Game on the Board

	cooperate	defect
cooperate	10,10	0,20
defect	20,0	5,5

Let's check if Grim trigger is an equilibrium:

$$10 + \beta \frac{10}{1-\beta} > 20 + \beta \frac{5}{1-\beta}$$

$$\frac{2}{3} < \beta < 1$$

Let's try another strategy.

Start in cooperative state. Cooperate unless last period was cooperative state and there was a deviation. If I deviated in that period, cooperate and then return to cooperative state. If I cooperated, deviate and then return to cooperative state. If we both deviated, cooperate and return to cooperative state.

I call this the repentance strategy. There are two key states to check for one-shot deviations from:

Cooperate in cooperative state. If I cooperate, and then go along with plan I get: $10+\beta\frac{10}{1-\beta}$. If I deviate today but then go along with plan, I am supposed to cooperate tomorrow while opponent deviates. I get 0. But then in the second period, we return to cooperation. The payoff is: $20+\beta(0)+\beta^2\frac{10}{1-\beta}$. One-shot deviation is not beneficial if:

$$10+\beta\frac{10}{1-\beta} > 20+\beta(0)+\beta^2\frac{10}{1-\beta}$$

$$10+\beta\frac{10}{1-\beta} > 20+\beta^2\frac{10}{1-\beta}$$

$$\frac{10}{1-\beta}\left(\beta-\beta^2\right) > 10$$

$$\frac{10}{1-\beta}\beta(1-\beta) > 10$$

$$10\beta > 10$$

Since $\beta < 1$, this is never optimal!