1 Proving the Utility Exists for Rational Prefrences

1.1 Finite X

For finite X: 1. U exists that represents $\succsim \Leftrightarrow \succsim$ is complete and transitive.
1.1. U exists that represents $\succsim \Rightarrow \succsim$ is complete and transitive.

$\geq$ on the reals is complete and transitive, then $\succsim$ must be complete and transitive.

1.2. U exists that represents $\succsim \Leftarrow \succsim$ is complete and transitive.

I claim that $U(x) = \#(\precsim(x))$ is such a utility function. We need to show that $\#(\precsim(x))$ represents $\succsim$.

To prove 1.2, we need to prove:

When $\succsim$ is complete and transitive: $\#(\precsim(x)) \geq \#(\precsim(y)) \Leftrightarrow x \succsim y$

1.2.1. $\#(\precsim(x)) \geq \#(\precsim(y)) \Rightarrow x \succsim y$

Re-write this by contrapositive.

Not $x \succsim y \Rightarrow$ Not $\#(\precsim(x)) \geq \#(\precsim(y))$

Since preferences are complete we can re-write this again. **We are left to prove.**

$y \succ x \Rightarrow \#(\precsim(y)) > \#(\precsim(x))$

By a corollary to containment:

$y \succ x \Leftrightarrow \precsim(x) \subset \precsim(y) \Rightarrow \#(\precsim(y)) > \#(\precsim(x))$

For finite sets:

$\precsim(x) \subset \precsim(y) \Rightarrow \#(\precsim(y)) > \#(\precsim(x))$

This is only true for finite sets. As a counter example: Integers $\mathbb{Z}$,primes $\mathbb{P}$.

$$\mathbb{P} \subset \mathbb{Z}$$

$$\#(\mathbb{P}) = \#(\mathbb{Z}) = \aleph_0$$

1.2.2. $\#(\precsim(x)) \geq \#(\precsim(y)) \Leftarrow x \succsim y$

By *containment.*

$x \succsim y \Leftrightarrow \precsim(y) \subseteq \precsim(x) \Rightarrow \#(\precsim(x)) \geq \#(\precsim(y))$

1.2 Countably Infinite X

$U(x) = \#(\precsim(x))$ may not work. Because $\#(\precsim(x))$ may be infinite.

Bowls of vanilla ice cream: $X = \{1, 2, ...\}$

Finn's preference are that he likes an even number of scoops to and odd number and otherwise prefers more ice cream.

$\precsim(2) = \{2,1,3,5,7,9....\}$

$U(2) = \aleph_0$

$U(4) = \aleph_0$

We need a way of getting around this.

Pick any arbitrary ordering on the bundles.

$x_1, x_2, ...$

$W(x_i) = \frac{1}{i^2}$

$U(x) = \sum_{\{y|y\in\precsim(x)\}} w(y)$

Let's go back to our ice cream example:

$\precsim(2) = \{2,1,3,....\}$

$\precsim(4) = \{2,4,1,3,....\}$

$4 \succ 2$

$U(2) = \frac{1}{2^2} + \sum_{i=0}^{\infty} \frac{1.0}{(2i+1)^2} = 1.4837$

$U(4) = \frac{1}{4^2} + \frac{1}{2^2} + \sum_{i=0}^{\infty} \frac{1.0}{(2i+1)^2} = 1.5462$

1.3 Uncountable X

Under uncountable X, there are rational $\succsim$ that have no utility representation.

Lexicographic Preference.

Cars are a bundle of horsepower, trunkspace. $(100,10)$ is 100 horsepower and 10 cubic feet of trunkspace.

I like a car strictly better if it has more horsepower, and otherwise if they have the same horse power, more trunk space. If the have the same number of both, they are indifferent.

$$(100,10) \succ (99,10)$$

$$(100,10) \succ (99,1000)$$

$$(100,10) \succ (100,9)$$

This is a rational preference relation.

There is no utility function that represents it.

Pick two real numbers $v_1 > v_2$

$$(v_1,1),(v_1,2)$$

$$(v_2, 1), (v_2, 2)$$

$$(v_1, 2) \succ (v_1, 1) \succ (v_2, 2) \succ (v_2, 1)$$

Suppose otherwise, there is some U that represents these preferences.

$$U(v_1, 2) > U(v_1, 1) > U(v_2, 2) > U(v_2, 1)$$

Since these are all real numbers we can construct two non-overlapping intervals.

$$[R_{v_1,2}, R_{v_1,1}]$$

$$[R_{v_2,2}, R_{v_2,1}]$$

Since the rationals are **dense in the reals,** within each of these intervals there is some rational number.

Find some rational in each of these intervals. For each possible real number v write the rational you find as $q(v)$.

This is a mapping from the reals onto the rationals.

Anytime you have a mapping from one set onto another, then the cardinality of the first set is weakly smaller.

$$\#(\mathbb{R}) \leq \#(\mathbb{Q})$$

This is a contradictn to the fact that the reals are bigger than the rationals.

$$\#(\mathbb{R}) = \aleph_1$$

$$\#(\mathbb{Q}) = \aleph_0$$

$$\#(\mathbb{R}) > \#(\mathbb{Q})$$