1 Homotheticity

$x \succsim y \Rightarrow tx \succsim ty$ for all $t \geq 0$.
A (smooth) utility function is homothetic if and only if the indifference curves are parallel along a rays through the origin.

$$MRS\left(x\right) = MRS\left(tx\right)$$

$$-\frac{\frac{\partial(u(x_1,x_2))}{\partial x_1}}{\frac{\partial(u(x_1,x_2))}{\partial x_2}} = -\frac{\frac{\partial(u(tx_1,tx_2))}{\partial x_1}}{\frac{\partial(u(tx_1,tx_2))}{\partial x_2}}$$

$u\left(x_1, x_2\right)$

$$u\left(2x_1, 2x_2\right) = u\left(x_1, x_2\right)$$

$$x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

$$-\frac{\frac{\partial\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)}{\partial x_1}}{\frac{\partial\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)}{\partial x_2}} = - \quad \frac{x_2}{x_1}$$

$$MRS\left(2x_1, 2x_2\right) = -\frac{2x_2}{2x_1} = -\frac{x_2}{x_1}$$

Since the MRS only depends on the ratio of x_1 and x_2, the utility function is homothetic.

$$u\left(tx_1, tx_2\right) = \left(tx_1\right)^{\frac{1}{2}} \left(tx_2\right)^{\frac{1}{2}} = t^{\frac{1}{2}} t^{\frac{1}{2}} \left(x_1\right)^{\frac{1}{2}} \left(x_2\right)^{\frac{1}{2}} = t\left(x_1\right)^{\frac{1}{2}} \left(x_2\right)^{\frac{1}{2}} = tu\left(x_1, x_2\right)$$

Here is another utility function with the same preferences but that has increasing returns to scale:

$$x_1 x_2$$

Because Homotheticity is a property of the preferences and not the utility function, it is an ordinal property and it is invariant to increasing transformations of the utility function.

2 Example

Suppose: $X = \{(2,0), (0,2), (4,0), (0,4), (1,1), (2,2)\}$.

Suppose preferences are **anti-symmetric** ($x \sim x$) but for all other pairs of distinct bundles either ($x \succ y$) or ($y \succ x$) and complete and transitive.

1. What can assume about $\succ$ if preferences are weakly monotonic?

$$(2,2) \succ (1,1)$$

1. What can assume about $\succ$ if preferences are strictly monotonic?

Suppose: $X = \{(2,0), (0,2), (4,0), (0,4), (1,1), (2,2)\}$.

$$(2,2) \succ (1,1), (4,0) \succ (2,0), (0,4) \succ (0,2), (2,2) \succ (0,2), (2,2) \succ (2,0)$$

2. Suppose $(2,0) \succ (0,2)$ and preferences are convex. What can we infer?

$$(1,1) \succ (0,2)$$

Suppose we assume preferences are weakly monotonic, convex, anti-symmetric, complete and transitive and that $(2,0) \succ (0,2)$.

$$(2,2) \succ (1,1) \succ (0,2)$$

By transitivity we can infer $(2,2) \succ (0,2)$. This preference is not ensured by weak monotonicity, we need to leverage the convexity and transitivity to get it.

3. Suppose we observe that $(2,0)$ is the choice from the set $\{(2,0), (1,1), (0,2)\}$ and preferences are complete, anti-symmetric, strictly monotonic, convex, and homothetic. What can we infer?

From monotonicity:

$$(2,2) \succ (1,1), (4,0) \succ (2,0), (0,4) \succ (0,2), (2,2) \succ (0,2), (2,2) \succ (2,0)$$

From the choice of $(2,0)$:

$$(2,0) \succ (1,1), (2,0) \succ (0,2)$$

From this and convexity:

$$(2,0) \succ (1,1) \succ (0,2)$$

From homotheticity:

$$(4,0) \succ (2,2) \succ (0,4)$$

What are the possible preference orderings?

This is impossible:

$$(2,0) \succ (1,1) \succ (0,2)$$

$$(4,0) \succ (2,2) \succ (0,4)$$

By strict monotonicity: $(2,2) \succ (2,0)$. Thus, $(4,0) \succ (2,2)$ have to be at the top of the ordering. Where can $(0,4)$ be? It has be be above $(0,2)$ but we don't know how it compares to $(1,1)$ or $(2,0)$. This gives the follow possibilities:

$$(4,0) \succ (2,2) \succ (0,4) \succ (2,0) \succ (1,1) \succ (0,2)$$

$$(4,0) \succ (2,2) \succ (2,0) \succ (0,4) \succ (1,1) \succ (0,2)$$

$$(4,0) \succ (2,2) \succ (2,0) \succ (1,1) \succ (0,4) \succ (0,2)$$

If we observe choice from $\{(0,4),(1,1)\}$ and $\{(0,4),(2,0)\}$ we will know the entire rank ordering.

3 Constrained Optimization - Some Intuition

Max u(x_1, x_2) subject to $p_1x_1 + p_2x_2 \leq m$

$$u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - m)$$