Linear Extension

Poset is a “Partially Ordered Set”, Partial Order

Poset, Partial Order: $\succsim$ reflexive, anti-symmetric, transitive

Linear Order: $\succsim$ reflexive, anti-symmetric, transitive, **complete**

A linear extension $\succsim_L$ is a Linear Order that is *consistent* with a partial order $\succsim$.

$\succsim \subseteq \succsim_L$

Partial order: $\succsim = \{(a,a),(b,b),(c,c),(a,b)\}$

Linear order: $\succsim = \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}$

Linear order: $\succsim = \{(a,a),(b,b),(c,c),(a,b),(c,b),(a,c)\}$

# 1 From Preferences to Choice

**Choice Function.**

A selection from every budget set. $\mathscr{B} = P(X)/\emptyset$

$C : \mathscr{B} \to \mathscr{B}$ such that $C(B) \subseteq B$.

**Finite Non-Emptyness:** For every finite $B \in \mathscr{B}$, $C(B) \neq \emptyset$.

$C(\{a,b\}) = \{a\}, C(\{a,b,c\}) = \{b\}$

**Coherence:**

For all $x, y$ and $B, B'$ such that $x, y \in B, B'$. $x \in C(B) \wedge y \notin C(B) \Rightarrow y \notin C(B')$

$C(\{a,b\}) = \{a,b\}, C(\{a,b,c\}) = \{b\}$

**Choice Function Induced By $\succsim$**

For $B \in \mathscr{B}$, $C_{\succsim}(B) = \{x | x \in B \wedge \forall y \in B, x \succsim y\}$

“The choice from $B$ are all the of the objects in $B$ that are a at least as good as every other object in $B$”

Example:

Linear order: $\succsim = \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}$

$C(\{a\}) = \{a\}, C(\{a,b\}) = \{a\}, C(\{a,b,c\}) = \{a\}$

Weak Order: $\succsim = \{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(a,c)\}$

$C(\{a\}) = \{a\}, C(\{a,b\}) = \{a,b\}$

Intransitive Order: $\succsim = \{(a,a),(b,b),(c,c),(a,b),(b,c),(c,a)\}$

This has a cycle of strict preferences $a \succ b \succ c \succ a$

$C(\{a\}) = \{a\}, C(\{a,b\}) = \{a\}, C(\{a,b,c\}) = \emptyset$

**Proposition 1. For a Finite $X$ and complete $\succsim$, Existence of an Empty Choice Is $\Leftrightarrow$ to a Cycle in $\succ$**

**Cycle in $\succ$ implies empty choice.**

The existence of a cycles implies there is some set of bundles $x_1, ..., x_n$ $x_1 \succ x_2 \succ ... \succ x_{n-1} \succ x_n \succ x_1$.

To show there is an empty choice, take $B = \{x_1, ..., x_n\}$

We need to show that for every $x_i \in B$ there is some $x_j \neq x_i$ such that $\neg (x_i \succsim x_j)$.

For $i > 1$, an $x_j$ meeting this condition is $x_{i-1}$ is such a bundle. Since $x_{i-1} \succ x_i$ by assumption, and $x_{i-1} \succ x_i$ implies $\neg (x_i \succsim x_{i-1})$ this condition is met for all $i > 1$.

For $i = 1$, note that $x_n \succ x_1$ which implies that $\neg (x_1 \succsim x_n)$ (This relies on the fact that $B$ is a loop).

For every $x_i \in B$, there is some bundle that $x_i$ is not at least as good as.

Let's prove the other direction:

**If there is an empty choice, then there is a cycle in $\succ$.**

If there is an empty choice then there is some budget $B$ such that $C(B) = \emptyset$.

For such a set, $\#(B) \geq 3$ because preferences are complete.

Choose any object in $B$. Call it $x_1$. Since $x_1$ in not in $C(B)$ and $\succsim$ is complete, there must be some $x_2 \in B$ such that $x_2 \succ x_1$. Now $x_2$ do the same exercise to find some $x_3 \succ x_2$. Continue to find some object better than $x_3$. If that is $x_1$ we have found a cycle. Otherwise continue.

Continue until we have constructred a chain $x_n \succ x_{n-1} \succ .... \succ x_1$. There must be something better than $x_n$, and that that thing has to already appear in the chain.

**If there is not a cycle, there is not an empty choice.**