Solution 1.

Let (x_1, y_1) and (x_2, y_2) be the draws. The expected cost of (x_1, y_1) is $(x_1 + y_1)$ conditional on $x_1 y_1 \geq x_2 y_2$. This cost is:

$$\int_0^1 \int_0^1 (x_1 + y_1) P(x_1 y_1 \geq x_2 y_2) \, dx_1 dy_1$$

Focusing on $P(x_1 y_1 \geq x_2 y_2)$:

$$P(x_1 y_1 \geq x_2 y_2) = \int_0^1 \int_0^{x_1 * y_1} dx_2 dy_2 + \int_{\frac{x_1 y_1}{x_2}}^1 \left(\int_{x_1 * y_1}^1 dx_2 \right) dy_2 = x_1 y_1 - x_1 y_1 log(x_1 y_1)$$

Thus, we can rewrite the cost of (x_1, y_1):

$$\int_0^1 \int_0^1 (x_1 + y_1)(x_1 y_1 - x_1 y_1 log(x_1 y_1)) \, dx_1 dy_1 = \frac{11}{18}$$

The expected cost of (x_2, y_2) is thus also $\frac{11}{18}$. This gives the total expected cost of $\frac{11}{9}$.

Solution 2.

For this solution, we calculate the $E_u(E(x + y|u))$. First, we focus on the inner expectation. Cost of a random bundle from indifference curve with utility u:

$$E(x + y|u(x, y) = u) = \int_u^1 f(x|u) \left(x + \frac{u}{x} \right) dx$$

$f(x|y) = \frac{f(x,u)}{f(u)}$. Let's focus on $f(u)$ first. Probability a randomly chosen bundle has $U(x, y) \leq u$.

$$F(u) = u + \int_u^1 \left(\int_0^{\frac{u}{x}} dy \right) dx = u - ulog(u)$$

Density for u:

$$f(u) = -log(u)$$

Now for $f(x, u)$. We have a tranformation of $f(x, x)$ with $x = x$ and $u = xy$. We need the inverse of the jacobian of this transformation which has partials $\begin{matrix} 1 & 0 \\ y & x \end{matrix}$. The detminant is x so the joint distribution of $f(x, u) = \frac{1}{x}$. Thus we have:

$$E(x + y|u(x, y) = u) = \int_u^1 \frac{1}{-xlog(u)} \left(x + \frac{u}{x} \right) dx = -\frac{2(1 - u)}{log(u)}$$

Let $F_2(u)$ be the CDF of the max of two draws from $F(u)$.

$$F_2(u) = (u - u log(u))^2$$

$$f_2(u) = 2u(log(u) - 1) log(u)$$

The expected cost of the best of two bundles is the expectation of $-\frac{2(1-u)}{log(u)}$ with density $f_2(u)$:

$$\int_0^1 \left(-\frac{2(1-u)}{log(u)}(2u(log(u) - 1) log(u))\right) du = \frac{11}{9}$$