Exercise. 6.1:

Suppose otherwise, then there is some bundle affordable under the convex combination of the budget sets: $x\left(tp+(1-t)\,p'\right)\leq tm+(1-t)\,m'$ but for which is not affordable under either budget set: $xp>m$ and $xp'>m'$. Note the first inequality can be written as:

$$txp+(1-t)\,xp'\leq tm+(1-t)\,m'$$

The next two inequalities can be written as:

$$txp>tm$$

$$(1-t)\,xp'>(1-t)\,m'$$

Adding the left of these and the right of these, we have:

$$txp+(1-t)\,xp'>tm+(1-t)\,m'$$

A contradiction.

Exercise. 6.4. Part C

Solution 1. *I think this is the simpler solution, so I present it first even thought it is not the first thing I tried.*

Let the randomly chosen bundles be $(x_1,y_1),(x_2,y_2)$

The average cost of the preferred bundle can be written:

$$\int_0^1\int_0^1(x_1+y_1)\,P\left(x_1y_1\geq x_2y_2\right)dx_1dy_1+\int_0^1\int_0^1(x_2+y_2)\,P\left(x_2y_2\geq x_1y_1\right)dx_2dy_2$$

Since (x_1,y_1) and (x_2,y_2) are independently and identically distributed, both of the terms in this sum have the same value, we can re-write it as:

$$=2\int_0^1\int_0^1(x_1+y_1)\,P\left(x_1y_1\geq x_2y_2\right)dx_1dy_1$$

Focusing on the $P\left(x_1y_1\geq x_2y_2\right)$ term:

$$P\left(x_1y_1\geq x_2y_2\right)=\int_0^1\int_0^{x_1*y_1}dx_2dy_2+\int_{\frac{x_1y_1}{x_2}}^1\left(\int_{x_1*y_1}^1dx_2\right)dy_2=x_1y_1-x_1y_1log\left(x_1y_1\right)$$

Plugging this into the expression $2\int_0^1\int_0^1(x_1+y_1)\,P\left(x_1y_1\geq x_2y_2\right)dx_1dy_1$ we get:

$$=2\int_0^1\int_0^1(x_1+y_1)\left(x_1y_1-x_1y_1log\left(x_1y_1\right)\right)dx_1dy_1=\frac{11}{9}$$

Solution 2.

When I wrote the question, this is the first thing I tried. It turns out to be a lot more complicated, but I present it anyway. For this solution, we take a very different approach. We first calculate the expected cost of a randomly chosen bundle on each indifference curve. Then we find the distribution over which distribution is the one that results from two random draws. We then take the expected cost over this distribution of u.

The cost of a random bundle from indifference curve with utility u is given by. Let $f(x|u)$ be the conditional density of x_1

$$E(x+y|u(x,y)=u)=\int_u^1 f(x|u)\left(x+\frac{u}{x}\right)dx$$

Conditional density $f(x|u)$ can be written this way: $f(x|u)=\frac{f(x,u)}{f(u)}$. Let's focus on $f(u)$ first. It is the derivative of $F(u)$ which is the probability a randomly chosen bundle has $U(x,y)\leq u$. This can be written:

$$F(u)=u+\int_u^1\left(\int_0^{\frac{u}{x}}dy\right)dx=u-ulog(u)$$

Taking the derivative gives us $f(u)$:

$$f(u)=-log(u)$$

Now for $f(x,u)$. We have a tranformation of $f(x,x)$ with $x=x$ and $u=xy$. We need the inverse of the jacobian of this transformation which has partials $\begin{matrix}1 & 0\\ y & x\end{matrix}$. The determinant is x so the joint distribution of $f(x,u)=\frac{1}{x}$. Thus we have:

$$f(x|u)=\frac{f(x,y)}{f(u)}=\frac{1}{x}\frac{1}{-log(u)}=\frac{1}{-xlog(u)}$$

Plugging this into: $E(x+y|u(x,y)=u)=\int_u^1 f(x|u)\left(x+\frac{u}{x}\right)dx$ we get:

$$E(x+y|u(x,y)=u)=\int_u^1\frac{1}{-xlog(u)}\left(x+\frac{u}{x}\right)dx=-\frac{2(1-u)}{log(u)}$$

Now that we know the expected cost on each indifference curve, we need to determine the distribution of which distribution results from the process of picking the best of two random bundles. Let $F_2(u)$ be the CDF of the max of two draws from $F(u)$. This is:

$$F_2(u)=(u-ulog(u))^2$$

Taking the derivative:

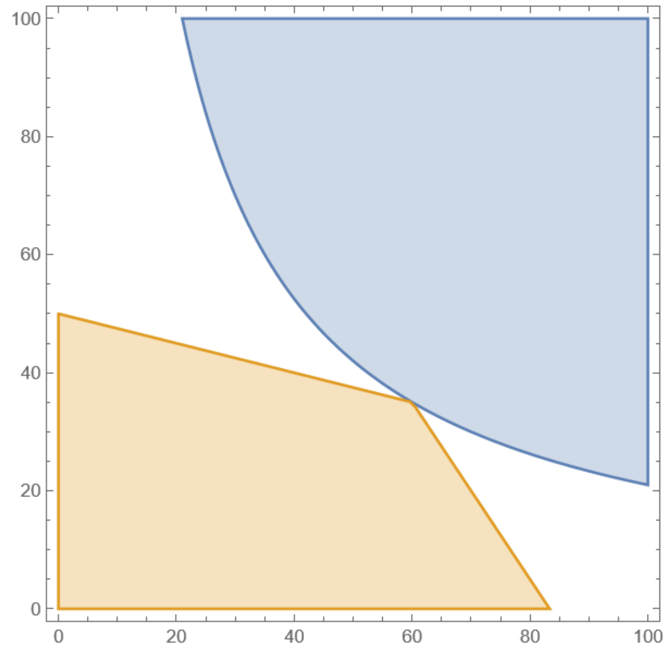
$$f_2(u)=2u(log(u)-1)log(u)$$

The expected cost of the best of two bundles can now be expressed as the expectation of $-\frac{2(1-u)}{log(u)}$ over u with density $f_2(u)$:

$$\int_0^1\left(-\frac{2(1-u)}{log(u)}(2u(log(u)-1)log(u))\right)du=\frac{11}{9}$$

Exercise. 7.1.

The optimum occurs where **both** constraints bind! The constraint is in orange below and the upper contour set of the optimal point $x_1=60, x_2=35$ is shown in blue.



The first order conditions are:

$$-\frac{\lambda_1}{2} - 3\lambda_2 + x_2 = 0$$

$$-2\lambda_1 - 2\lambda_2 + x_1 = 0$$

Plugging in the optimum:

$$\lambda_1 = 22, \lambda_2 = 8$$

Thus, constraint 1 relaxing constraint 1 improves utility at a faster rate than constraint 2.

Exercise. 7.2

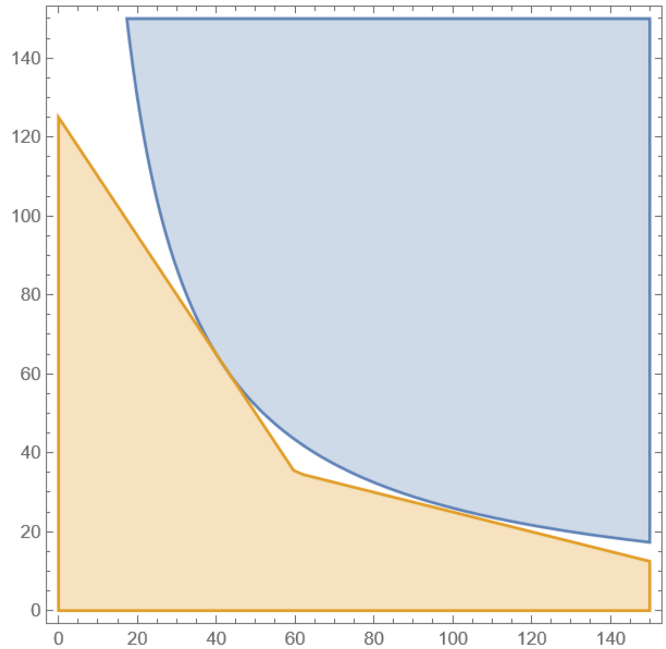
Now we need only worry about one constraint being met at a time and can ignore whether the other is met.

If constraint 1 binds, the optimal solution is: $x_1 = 100, x_2 = 25$ which gives utility 2500. Notice this solution does not meet constraint 2, but since we have an **or** constraint, it doesn't matter.

If constraint 2 binds, the optimal solution is: $x_1 = \frac{125}{3}, x_2 = \frac{125}{2}$ which gives utility $\frac{15625}{6} > 2500$. Notice this solution does not meet constraint 1, but since we have an **or** constraint, it doesn't matter.

Thus, the solution is $\left(\frac{125}{3}, \frac{125}{2}\right)$.

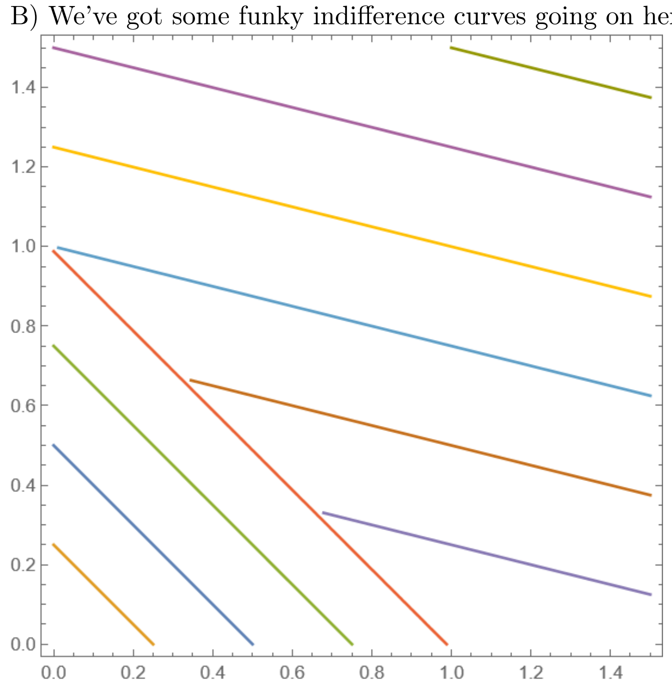
Here is the situation depicted graphically (the second constraint is the steeper one). The blue area is the upper contour set of $\left(\frac{125}{3}, \frac{125}{2}\right)$.



Exercise. 7.3.

A) The preferences are not continuous. For instance, the set $\precsim (0.5, 0, 5)$ is the union of the set of bundles that have $x_1 + x_2 \geq 1$ and $x_1 + 4x_2 \geq 2.5$ and the set of bundles where $x_1 + x_2 < 1$. This set has an open region along the $x_1 + x_2 == 1$ to the "left" of $(0.5, 0.5)$.

B) We've got some funky indifference curves going on here:



C) Consume only good 1, $x_1 = \frac{1}{p_1}$.

D) $x_1 = \frac{1}{2-p_1}, x_2 = \frac{1-p_1}{2-p_1}$

E) As p_1 increases, demand for x_1 increases in the range $p_1 \in (0, 1)$. Demand is "Giffen".