Econ 8100 - Final

warm-up.

A consumer has utility function $(x_1^\alpha + x_2^\alpha)^\beta$. Income is m and prices are p_1, p_2.

A. Set up the Lagrangian that turns the consumer's constrained utility maximization problem into an unconstrained maximization. Use λ for the Lagrange multiplier on the budget constraint.

$$(x_1^\alpha + x_2^\alpha)^\beta - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)$$

B. Write down the first order conditions for this unconstrained maximization.

$$\frac{\partial\left((x_1^\alpha + x_2^\alpha)^\beta - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)\right)}{\partial x_1} = 0$$

$$\frac{\partial\left((x_1^\alpha + x_2^\alpha)^\beta - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)\right)}{\partial x_2} = 0$$

$$\alpha\beta x_1^{\alpha-1}\left(x_1^\alpha + x_2^\alpha\right)^{\beta-1} = \lambda p_1$$

$$\alpha\beta x_2^{\alpha-1}\left(x_1^\alpha + x_2^\alpha\right)^{\beta-1} = \lambda p_2$$

C. Suppose $\alpha = 2, \beta = \frac{1}{2}$. How does this consumer's utility change when one of the goods is increases marginally from a point where $x_1 = x_2 = x$?

As x increases utility increases at the rate $\sqrt{2}$ as x is increased.

$$\frac{\partial u\left(x_1, x_2\right)}{x_i} = \frac{x_i}{\sqrt{x_1^2 + x_2^2}} = \frac{x}{\sqrt{x^2 + x^2}} = \frac{1}{\sqrt{2}}$$

D. Suppose $\alpha = 2, \beta = \frac{1}{2}, p_1 = p_2$. Solve for λ from part **B.**

$$x_1\left(x_1^2 + x_2^2\right)^{-\frac{1}{2}} = \lambda p$$

$$x_2\left(x_1^2+x_2^2\right)^{-\frac{1}{2}}=\lambda p$$

Combined:

$$x_1=x_2=x$$

Solving for λ :

$$x\frac{1}{\left(2x^2\right)^{\frac{1}{2}}}=\lambda p$$

$$x\frac{1}{\sqrt{2}x}=\lambda p$$

$$\frac{1}{p\sqrt{2}}=\lambda$$

E. Interpret the number you found in **D** using your answer to **C.**
λ is the marginal utility on money at the optimum. Increasing either good will increase utility marginally by $\frac{1}{\sqrt{2}}$. As income increases, the marginal amount of either good that can be afforded is $\frac{1}{p}$. Thus, as money increases, the marginal increase in utility is $\frac{1}{p}\frac{1}{\sqrt{2}}$.

cournot.

J firms compete in Cournot oligopoly. Each has cost function $c\left(q_i\right)=q_i+a$. Consumer demand is $q\left(p\right)=\frac{b}{p}$.

A. What are the firm's profit functions as a function of q_i and q_{-i}.

$$\frac{b}{q_i+q_{-i}}q_i-\left(q_i+a\right)$$

B. What is firm i's best response to quantity q_{-i} of the other firms?

The FOC of the profit function is:

$$\frac{\partial\left(\frac{b}{q_i+q_{-i}}q_i-\left(q_i+a\right)\right)}{\partial q_i}=0$$

$$\frac{b}{q_{-i}+q_i}-\frac{bq_i}{\left(q_{-i}+q_i\right)^2}-1=0$$

$$\frac{b\left(q_{-i}+q_i\right)}{\left(q_{-i}+q_i\right)^2}-\frac{bq_i}{\left(q_{-i}+q_i\right)^2}-1=0$$

$$\frac{bq_{-i} + bq_i}{(q_{-i} + q_i)^2} - \frac{bq_i}{(q_{-i} + q_i)^2} = 1$$

$$\frac{bq_{-i} + bq_i - bq_i}{(q_{-i} + q_i)^2} = 1$$

$$\frac{bq_{-i}}{(q_{-i} + q_i)^2} = 1$$

$$bq_{-i} = (q_{-i} + q_i)^2$$

$$\sqrt{bq_{-i}} = q_{-i} + q_i$$

$$q_i = \sqrt{bq_{-i}} - q_{-i}$$

C. Find the symmetric Nash equilibrium.

$$q = \sqrt{b(J-1)q} - (J-1)q$$

$$Jq = \sqrt{b(J-1)q}$$

$$J^2 q^2 = b(J-1)q$$

$$q = \frac{b(J-1)}{J^2}$$

D. Show that as J increases, the equilibrium price approaches the firm's marginal cost from above.

The total supply is:

$$Q(J) = J\frac{b(J-1)}{J^2} = \frac{b(J-1)}{J}$$

This is increasing in J and:

$$Lim_{J\to\infty}\left(\frac{b(J-1)}{J}\right) = b$$

Using the inverse demand:

$$p = \frac{b}{Q(J)}$$

This is decreasing in Q which is increasing in J thus, p is decreasing in J and:

$$lim_{J\to\infty}\left(\frac{b}{Q(J)}\right) = \frac{b}{Lim_{J\to\infty}Q(J)} = 1$$

The marginal cost of each firm is:

$$\frac{\partial(c(q_i))}{\partial q_i} = \frac{\partial(q_i + a)}{\partial q_i} = 1$$

Thus, the price approaches marginal cost from above.

E. In terms of a and b, what is the number of firms J such that profit of each firm is zero?

Profit in equilibrium is:

$$\pi(J) = \frac{b}{\frac{b(J-1)}{J}}\left(\frac{b(J-1)}{J^2}\right) - \left(\frac{b(J-1)}{J^2} + a\right)$$

Set this to zero:

$$\frac{b}{\frac{b(J-1)}{J}}\left(\frac{b(J-1)}{J^2}\right) - \left(\frac{b(J-1)}{J^2} + a\right) = 0$$

Solve for J :

$$\frac{J}{J-1}\left(\frac{b(J-1)}{J^2}\right) - \left(\frac{b(J-1)}{J^2} + a\right) = 0$$

$$\left(\frac{b}{J}\right) - \frac{b(J-1)}{J^2} = a$$

$$\frac{bJ}{J^2} - \frac{b(J-1)}{J^2} = a$$

$$\frac{bJ}{J^2} - \frac{bJ-b}{J^2} = a$$

$$\frac{b}{J^2} = a$$

$$J = \left(\frac{a}{b}\right)^{\frac{1}{2}}$$

technologies.

A firm has access to two technologies: $f(x_1, x_2) = 2x_1^{\frac{1}{8}} x_2^{\frac{1}{8}}$ and $g(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$. Suppose $w_1 = w_2 = 1$.

A. Show the cost function for f is of the form $c(1)\, y^{\alpha}$ where α is the reciprocal of the degree of homogeneity of f.

Set up the lagrangian for cost minimization:

$$x_1 + x_2 + \mu \left(y - 2x_1^{\frac{1}{8}} x_2^{\frac{1}{8}}\right)$$

The FOCs:

$$\frac{\partial \left(x_1 + x_2 + \mu \left(y - 2x_1^{\frac{1}{8}} x_2^{\frac{1}{8}}\right)\right)}{\partial x_1} = 0$$

$$\frac{\partial \left(x_1 + x_2 + \mu \left(y - 2x_1^{\frac{1}{8}} x_2^{\frac{1}{8}}\right)\right)}{\partial x_2} = 0$$

$$1 = \frac{\mu \sqrt[8]{x_2}}{4x_1^{7/8}}$$

$$1 = \frac{\mu \sqrt[8]{x_1}}{4x_2^{7/8}}$$

Solving these:

$$x_1 = x_2 = x$$

Plug this back into the production constraint:

$$y = 2x^{\frac{1}{8}} x^{\frac{1}{8}}$$

$$\frac{y}{2} = x^{\frac{1}{4}}$$

$$x = \left(\frac{y}{2}\right)^4 = \frac{y^4}{16}$$

Cost is:

$$c(y) = \frac{y^4}{16} + \frac{y^4}{16} = \frac{y^4}{8}$$

The cost of one unit is:

$$c(1) = \frac{1}{8}$$

Thus:

$$c(y) = \frac{1}{8}y^4 = c(1)\,y^4$$

The degree of homogeneity of this production function is $\frac{1}{4}$. Thus, we have the desired form.

B. If the firm can freely choose between these technologies, what is it's cost function?

The cost function associated with f is:

$$c_f(y) = \frac{1}{8}y^4$$

Using cost minimization on g and knowing it will have the homogeneity form:

$$c_g(y) = 2y^2$$

The firm will choose the cheaper technology. Let's see when g is cheaper:

$$c_g(y) < c_f(y)$$

$$2y^2 < \frac{1}{8}y^4$$

$$16y^2 < y^4$$

$$16 < y^2$$

$$4 < y$$

Thus, the cost function is:

$$c(y) = \begin{cases} \frac{y^4}{8} & y \leq 4 \\ 2y^2 & y > 4 \end{cases}$$

C. What quantity does a price-taking firm produce if the price is $p = 32$.

Don't let the above work distract you. Set up the profit functions of either technology:

$$\pi_f(y) = 32y - \frac{y^4}{8}$$

$$\pi_g(y) = 32y - 2y^2$$

Maximize both. The FOCs:

For f:

$$\frac{\partial\left(32 * y - \frac{y^4}{8}\right)}{\partial y} = 0$$

$$32 = \frac{y^3}{2}$$

$$4 = y$$

For g:

$$\frac{\partial\left(32 * y - 2y^2\right)}{\partial y} = 0$$

$$32 = 4y$$

$$8 = y$$

Profit for f:

$$\pi_f(4) = 32(4) - \frac{(4^4)}{8} = 96$$

Profit for g:

$$\pi_g(8) = 32(8) - 2\left(8^2\right) = 128$$

Use g profit is 128!

D. Is the firms cost function from part **B** concave in y? Is it quasi-concave in y? Justify your answer.

Is it concave? *No way*, for instance when $y < 4$ the function is: $\frac{y^4}{8}$ which is strictly convex. However, the cost function is monotonic so it is quasi-concave.

E. Prove that the profit function is quasi-concave in x_1, x_2 for price-taking firms with the production function g.

Oof. Let's set it up:

$$px_1^{\frac{1}{4}}x_2^{\frac{1}{4}} - x_1 - x_2$$

$px_1^{\frac{1}{4}}x_2^{\frac{1}{4}}$ is a monotonic transformation of $\frac{1}{4}ln(x_1) + \frac{1}{4}ln(x_2)$ which is a sum of concave functions and thus concave. Since it is monotonic transformation of a concave function it is quasi-concave. It is also homogeneous of degree less than 1 (degree $\frac{1}{2}$). Since it is a quasi-concave function homogeneous of degree less than 1 it is concave. *Of course, there are more traditional ways to show this.*

Since $-x_1$ and $-x_2$ are linear, they are concave.

Thus, the profit function is a sum of concave functions and is concave.