A Binomial Tail Inequality for Successes

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Abstract

I provide a monotonicity result on binomial tail probabilities in terms of the number of successes. Consider two binomial processes with n trials. For any $k \in 1, ..., n-1$, as long as the expected number of successes in the first process is at least $\frac{n}{n-1}(k-1)$ and the expected number of successes in the second process is at least $\frac{k}{k-1}$ times larger than that of the first, then the probability of $k-1$ or fewer successes in the first process is strictly larger than the probability of k or fewer successes in the second.

1 Introduction

This binomial distribution is a fundamental probability distribution.

(1)
$$b(k;n,p) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, ..., n$$
$$b(k;,n,p) = 0, \quad k = n+1, ...$$

The result in this paper regards the CDF (tail probability) of the binomial:

(2)
$$B(k;n,p) = \sum_{i=0}^{k} b(i;n,p), \quad k = 0, 1, ..., n$$

Inequalities regarding tail probabilities of the binomial distribution and more general heterogeneous sums of Bernoulli random variables have been studied in Hoeffding [1956], Gleser [1975], Boland and Proschan [1983], Xu and Balakrishnan [2011]. In particular, Anderson and Samuels [1967] prove several useful monotonicity results for $B(k;n,p)$ for the number of trials n (with a fixed number of successes k) as well as joint monotonicity in n and k. *Theorem 2.3* of Anderson and Samuels [1967] states:

(3)
$$B\left(k;n,\frac{\lambda}{n}\right) > B\left(k;n-1,\frac{\lambda}{n-1}\right), \quad k < \lambda, n \geq \frac{\lambda}{\lambda - k}$$

(4)
$$B\left(k;n,\frac{\lambda}{n}\right) < B\left(k;n-1,\frac{\lambda}{n-1}\right), \quad k > \lambda$$

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Theorem 2.2 states:

(5)
$$B\left(k;n,\frac{\lambda}{n}\right) > B\left(k-1;n-1,\frac{\lambda-1}{n-1}\right), \quad k \leq \lambda - 1$$

(6)
$$B\left(k;n,\frac{\lambda}{n}\right) < B\left(k-1;n-1,\frac{\lambda-1}{n-1}\right), \quad k \geq \lambda$$

Anderson and Samuels [1967] uses these inequalities to derive results about the monotonicity of the error in approximating binomial tail probabilities using the Poisson distribution. This paper focuses instead on monotonicity in k with fixed n. The result is stated here, in notation consistent with Anderson and Samuels [1967].

Proposition 1. $B\left(k-1;n,\frac{k-1}{k}\frac{\lambda}{n}\right) > B\left(k;n,\frac{\lambda}{n}\right)$, $\lambda \in \left[\frac{n}{n-1}k, n\right]$, $k \in \{1,...,n-1\}$.

Proposition 1 can be interpreted in terms of system reliability [see also: Boland and Proschan, 1983]. Suppose a redundant system has n identical components. Initially these components are sufficiently reliable so that the expected number of components functioning over some period of time is at least $\frac{n}{n-1}(k-1)$ for some $k \in \{1,...,n\}$. In this system, suppose there is a 95% chance that strictly more than $k-1$ components are functioning in the same time-span. Call $k-1$ the 95% *reliability rating* of the system. Now suppose an engineer needs to upgrade the components so that the 95% *reliability rating* is k. The engineer wants to increase the 95% *reliability rating* of the system by a factor of $\frac{k}{k-1}$. Proposition 1 states that increasing the reliability of each component proportionally (by $\frac{k}{k-1}$) is more than enough to accomplish this task.

A few simple calculations show that the relevant interval on λ for which 1 applies cannot be extended to $\lambda \in [k, 1]$. For instance: $B\left(2;5,\frac{2}{5}\right) > B\left(3;5,\frac{3}{5}\right) < B\left(4;5,\frac{4}{5}\right)$. Of course, as n grows large, the lower bound of the interval approaches k. In fact, taking $\lambda = \frac{n}{n-1}k$, note that as $n \to \infty$, the binomial random variable approaches a Poisson random variable R_k with mean k. Poisson random variables have a analogous monotonicity property: $P\left(R_{k-1} \leq k-1\right) > P\left(R_k \leq k\right)$ [Adell and Jodra, 2005, Lemma 1]. The proposition here demonstrates that this particular sequence of binomial random variables converging to R_{k-1} and R_k maintains this monotonicity for finite n.

2 Proof

The proof of the proposition is broken into several parts. Let $g\left(k;n,\frac{\lambda}{n}\right) \equiv B\left(k-1;n,\frac{k-1}{k}\frac{\lambda}{n}\right) - B\left(k;n,\frac{\lambda}{n}\right)$. The proposition is equivalent to $g\left(k;n,\frac{\lambda}{n}\right) > 0$, $\lambda \in \left[\frac{n}{n-1}k, n\right]$, $k \in \{1,...,n-1\}$. I begin by proving the proposition holds at the lower bound of the interval (lemma 2). Since $g\left(k-1;n,\frac{n}{n}\right) = B\left(k-1;n,\frac{k-1}{k}\right) - B\left(k-1;n,1\right) = B\left(k-1;n,\frac{k-1}{k}\right) > 0$ it also holds at the upper bound. The final step is to demonstrate that $g\left(k-1;n,\frac{\lambda}{n}\right)$ is quasi-concave in λ on $\lambda \in \left[\frac{n}{n-1}k, n\right]$ (lemma 2). That is, it is either increasing or increasing up to a point and then decreasing. In either case, $g\left(k-1;n,\frac{\lambda}{n}\right)$ must remain strictly positive over $\lambda \in \left[\frac{n}{n-1}k, n\right]$, proving the proposition.

Lemma 2. $B\left(k-1;n,\frac{k-1}{k}\frac{\lambda}{n}\right) > B\left(k;n,\frac{\lambda}{n}\right)$, $\lambda = \frac{n}{n-1}k$, $n > 1$, $k \in \{1,...,n-1\}$

Proof. Let $X_{p_1}, X_{p_2}, ..., X_{p_{n+1}}$ be a sequence of $n+1$ independent Bernoulli random variables each with parameter p_i and let $\mathbf{p} = (p_1, p_2, ..., p_{n+1})$. Define:

$$H(k; \mathbf{p}) = P\left(\sum_{i=1}^{n+1} X_{p_i} \leq k\right) \tag{7}$$

Define the following parameterized $n+1$ length vectors which generate two sequences of heterogeneous independent Bernoulli random variables:

$$\mathbf{p}_z = \left(1 - z, \frac{k-1}{n-1} + \frac{z}{n}, ..., \frac{k-1}{n-1} + \frac{z}{n}\right) \tag{8}$$

$$\tilde{\mathbf{p}}_z = \left(z, \frac{k}{n-1} - \frac{z}{n}, ..., \frac{k}{n-1} - \frac{z}{n}\right) \tag{9}$$

The lemma is equivalent to the following:

$$H(k; \mathbf{p}_0) > H(k; \tilde{\mathbf{p}}_0) \tag{10}$$

Let $a_{k-1} = \frac{n(n-k)}{(n-1)(n+1)}$ and $a_k = \frac{nk}{(n-1)(n+1)}$. $\mathbf{p}_{a_{k-1}}$ and $\tilde{\mathbf{p}}_{a_k}$ are both homogeneous sequences of Bernoulli random variables with, respectively probabilities of success a_{k-1} and a_k. Since $B(k; n, p)$ is decreasing in p and $1 - a_{k-1} < a_k$ for $n > 1$.

$$H\left(k; \mathbf{p}_{a_{k-1}}\right) = B(k; n+1, 1 - a_{k-1}) > B(k; n+1, a_k) = H\left(k; \tilde{\mathbf{p}}_{a_k}\right) \tag{11}$$

Combining 10 and 11, a sufficient condition for the lemma is:

$$H(k; \mathbf{p}_0) \geq H\left(k; \mathbf{p}_{\mathbf{a_{k-1}}}\right) \;\&\; H\left(k; \tilde{\mathbf{p}}_{a_k}\right) \geq H(k; \tilde{\mathbf{p}}_0) \tag{12}$$

This is true if:

$$\frac{\partial H(k; \mathbf{p}_z)}{\partial z} \leq 0, \;\; z \in [0, a_{k-1}] \tag{13}$$

$$\frac{\partial H(k; \tilde{\mathbf{p}}_z)}{\partial z} \geq 0, \;\; z \in [0, a_k] \tag{14}$$

$H(k; \mathbf{p}_z)$ and $H(k; \tilde{\mathbf{p}}_z)$ can be written [See Anderson and Samuels, 1967, page 3]:

$$H\left(k;\mathbf{p}_z\right)=\left(1-z\right)B\left(k-1;n,\frac{k-1}{n-1}+\frac{z}{n}\right)+zB\left(k;n,\frac{k-1}{n-1}+\frac{z}{n}\right) \tag{15}$$

$$H\left(k;\tilde{\mathbf{p}}_z\right)=zB\left(k-1;n,\frac{k}{n-1}-\frac{z}{n}\right)+\left(1-z\right)B\left(k;n,\frac{k}{n-1}-\frac{z}{n}\right) \tag{16}$$

The derivatives with respect to z are:

$$\frac{\partial H\left(k;\mathbf{p}_z\right)}{\partial z}=b\left(k;n,\frac{k-1}{n-1}+\frac{z}{n}\right)+\left(1-z\right)\frac{\partial B\left(k-1;n,\frac{k-1}{n-1}+\frac{z}{n}\right)}{\partial z}+z\frac{\partial B\left(k;n,\frac{k-1}{n-1}+\frac{z}{n}\right)}{\partial z} \tag{17}$$

$$\frac{\partial H\left(k;\mathbf{p}_z\right)}{\partial z}=-b\left(k;n,\frac{k}{n-1}-\frac{z}{n}\right)+z\frac{\partial B\left(k-1;n,\frac{k}{n-1}-\frac{z}{n}\right)}{\partial z}+\left(1-z\right)\frac{\partial B\left(k;n,\frac{k}{n-1}-\frac{z}{n}\right)}{\partial z} \tag{18}$$

The derivative of a binomial CDF $B\left(k;n,p\right)$ with respect to the probability of success p is:

$$\begin{aligned}\frac{\partial B\left(k;n,p\right)}{\partial p}&=\left(n-k\right)\binom{n}{k}\frac{\partial\int_0^{1-p}t^{n-k-1}\left(1-t\right)^k dt}{\partial p}=\\&-\left(n-k\right)\binom{n}{k}\left(1-p\right)^{n-k-1}p^k=-nb\left(k;n-1,p\right)\end{aligned} \tag{19}$$

Thus, 17 and 18 can be written:

$$\frac{\partial H\left(k;\mathbf{p}_z\right)}{\partial z}=b\left(k;n,\frac{k-1}{n-1}+\frac{z}{n}\right)-\left(1-z\right)b\left(k-1;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)-zb\left(k;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right) \tag{20}$$

$$\frac{\partial H\left(k;\mathbf{p}_z\right)}{\partial z}=-b\left(k;n,\frac{k}{n-1}-\frac{z}{n}\right)+zb\left(k-1;n-1,\frac{k}{n-1}-\frac{z}{n}\right)+\left(1-z\right)b\left(k;n-1,\frac{k}{n-1}-\frac{z}{n}\right) \tag{21}$$

Using these expressions, the sufficient conditions 13 and 14 can be re-written:

$$\begin{aligned}&b\left(k;n,\frac{k-1}{n-1}+\frac{z}{n}\right)\leq\\&\left(1-z\right)b\left(k-1;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)+zb\left(k;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right),\ z\in\left[0,a_{k-1}\right]\end{aligned} \tag{22}$$

$$b\left(k;n,\frac{k}{n-1}-\frac{z}{n}\right)\leq zb\left(k-1;n-1,\frac{k}{n-1}-\frac{z}{n}\right)+(1-z)\,b\left(k;n-1,\frac{k}{n-1}-\frac{z}{n}\right),\ \ z\in[0,a_k] \tag{23}$$

$b\left(k;n,\frac{k-1}{n-1}+\frac{z}{n}\right)$ and $b\left(k;n,\frac{k}{n-1}-\frac{z}{n}\right)$ can be re-written:

$$b\left(k;n,\frac{k-1}{n-1}+\frac{z}{n}\right)=\left(\frac{k-1}{n-1}+\frac{z}{n}\right)b\left(k-1;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)+\left(1-\frac{k-1}{n-1}-\frac{z}{n}\right)b\left(k;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right) \tag{24}$$

$$b\left(k;n,\frac{k}{n-1}-\frac{z}{n}\right)=\left(\frac{k}{n-1}-\frac{z}{n}\right)b\left(k-1;n-1,\frac{k}{n-1}-\frac{z}{n}\right)+\left(1-\frac{k}{n-1}+\frac{z}{n}\right)b\left(k;n-1,\frac{k}{n-1}-\frac{z}{n}\right) \tag{25}$$

Substituting these into inequalities 22 and 23 provides the following sufficient conditions for the lemma:

$$\left(\frac{k-1}{n-1}+\frac{z}{n}\right)b\left(k-1;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)+\left(1-\frac{k-1}{n-1}-\frac{z}{n}\right)b\left(k;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)\leq (1-z)\,b\left(k-1;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)+zb\left(k;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right),\ \ z\in[0,a_{k-1}] \tag{26}$$

$$\left(\frac{k}{n-1}-\frac{z}{n}\right)b\left(k-1;n-1,\frac{k}{n-1}-\frac{z}{n}\right)+\left(1-\frac{k}{n-1}+\frac{z}{n}\right)b\left(k;n-1,\frac{k}{n-1}-\frac{z}{n}\right)\leq zb\left(k-1;n-1,\frac{k}{n-1}-\frac{z}{n}\right)+(1-z)\,b\left(k;n-1,\frac{k}{n-1}-\frac{z}{n}\right),\ \ z\in[0,a_k] \tag{27}$$

The left and right side of conditions 26 and 27 are both weighted averages of two binomial probabilities from the same process. In each case, the right side puts at least as much weight on the larger of the two probabilities. The right side of inequality 26 puts more weight on $b\left(k-1;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)$ since $(1-z)\geq\left(\frac{k-1}{n-1}+\frac{z}{n}\right)$ when $z\in[0,a_{k-1}]$. This inequality can be rewritten as $1-\frac{n+1}{n}z\geq\frac{k-1}{n-1}$. Since the left is decreasing in z, it is sufficient to check the inequality for $z=a_{k-1}=\frac{n(n-k)}{(n-1)(n+1)}$ which yields $\frac{k-1}{n-1}\geq\frac{k-1}{n-1}$. The right side of inequality 27 puts more weight on $b\left(k;n,\frac{k}{n-1}-\frac{z}{n}\right)$ since $(1-z)\geq\left(1-\frac{k}{n-1}+\frac{z}{n}\right)$ when $z\in[0,a_k]$. Note this inequality can be rewritten $\left(1-\frac{n+1}{n}z\right)\geq\left(1-\frac{k}{n-1}\right)$. Since the left is decreasing in z it is sufficient to check the inequality for $z=a_k=\frac{nk}{(n-1)(n+1)}$ which yields $\left(1-\frac{k}{n-1}\right)\geq\left(1-\frac{k}{n-1}\right)$. Thus, these inequalities are true if:

$$b\left(k-1;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)\geq b\left(k;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right),\ \ z\in[0,a_{k-1}] \tag{28}$$

$$b\left(k;n-1,\frac{k}{n-1}-\frac{z}{n}\right)\geq b\left(k-1;n-1,\frac{k}{n-1}-\frac{z}{n}\right),\ \ z\in[0,a_k] \tag{29}$$

Note that a mode of the binomial distribution is $\lfloor(n+1)\,p\rfloor$. For inequality 26, $k>n\left(\frac{k-1}{n-1}+\frac{z}{n}\right)>k-1$ for $z\in\left[0,\frac{n-k}{n-1}\right]$ when $n>k$. Since $a_{k-1}=\frac{n(n-k)}{(n-1)(n+1)}$ and $\frac{n-k}{n-1}<\frac{n(n-k)}{(n-1)(n+1)}$, the mode of a binomial distribution with $n-1$ trials and probability of success $\frac{k-1}{n-1}+\frac{z}{n}$ is $k-1$ on $z\in[0,a_{k-1}]$. Thus, $b\left(k-1;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)\geq b\left(k;n-1,\frac{k-1}{n-1}+\frac{z}{n}\right)$, $z\in[0,a_{k-1}]$. For inequality 27, $k\leq n\left(\frac{k}{n-1}-\frac{z}{n}\right)\leq k+1$ for $z\in\left[0,\frac{k}{n-1}\right]$ when $n>k$. Since $a_k=\frac{nk}{(n-1)(n+1)}<\frac{k}{n-1}$ the mode of a binomial distribution with $n-1$ trials and probability of success $\frac{k}{n-1}-\frac{z}{n}$ is k on $z\in[0,a_k]$. This proves 28, 29 and confirms the sufficient conditions for the lemma given in 26 and 27. ☐

Lemma 3. *$g\left(k-1;n,\frac{\lambda}{n}\right)$ is quasi-concave in λ on $\lambda\in\left[\frac{n}{n-1}k,n\right]$.*

Proof. This is equivalent to the quasi-concavity of $\tilde{g}\,(k;n,p)\equiv B\left(k-1;n,\frac{k-1}{k}p\right)-B\,(k;n,p)$ on $p\in\left[\frac{k}{n-1},1\right]$. Differentiating $\tilde{g}\,(k;n,p)$ with respect to p:

$$\frac{\partial\tilde{g}\,(k;n,p)}{\partial p}=n\binom{n-1}{k}p^k\,(1-p)^{n-k-1}-n\left(\frac{k-1}{k}\right)\binom{n-1}{k-1}\left(\frac{k-1}{k}p\right)^{k-1}\left(1-\frac{k-1}{k}p\right)^{n-k} \tag{30}$$

Thus, $\tilde{g}\,(k;n,p)$ is increasing if:

$$\frac{p}{1-p}\left(\frac{1-p}{1-\frac{k-1}{k}p}\right)^{n-k}\geq\frac{k-1}{n-k}\left(\frac{k-1}{k}\right)^{k-1} \tag{31}$$

Quasi-concavity follows from the fact that the left side of this inequality is decreasing over $p\in\left[\frac{k}{n-1},1\right]$ and the inequality is true at $p=\frac{k}{n-1}$. The left side is decreasing since the derivative is with respect to p is negative when $p\in\left[\frac{k}{n-1},1\right]$:

$$\frac{1}{(1-p)^2}\left(\frac{1-p}{1-\frac{k-1}{k}p}\right)^{n-k}-(n-k)\,\frac{p}{1-p}\left(\frac{1-p}{1-\frac{k-1}{k}p}\right)^{n-k-1}\frac{1-\frac{k-1}{k}}{\left(1-\frac{k-1}{k}p\right)^2}\leq 0\Leftrightarrow p\geq\frac{k}{n-1}$$

Plugging $p=\frac{k}{n-1}$ into 31 yields:

$$\frac{k}{n-1-k}\left(\frac{n-1-k}{n-k}\right)^{n-k}\geq\frac{k-1}{n-k}\left(\frac{k-1}{k}\right)^{k-1} \tag{32}$$

This can be rearranged into the following inequality involving a weighted geometric mean:

$$\left(\frac{k}{k-1}\right)^{\frac{k}{n-1}}\left(\frac{n-k-1}{n-k}\right)^{\frac{n-k-1}{n-1}}\geq 1 \tag{33}$$

This is true by the harmonic-geometric means inequality:

$$\left(\frac{k}{k-1}\right)^{\frac{k}{n-1}}\left(\frac{n-k-1}{n-k}\right)^{\frac{n-k-1}{n-1}}\geq\left(\frac{k}{n-1}\frac{k-1}{k}+\frac{n-k-1}{n-1}\frac{n-k}{n-k-1}\right)^{-1}=1 \tag{34}$$

Thus $\tilde{g}\left(k;n,p\right)$ is increasing at $p=\frac{k}{n-1}$. Since the left of 31 decreases over $p\in\left[\frac{k}{n-1},1\right]$, $\tilde{g}\left(k;n,p\right)$ is either increasing up to a point and then decreasing or is increasing over all of $p\in\left[\frac{k}{n-1},1\right]$. It is thus quasi-concave. $\square$

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